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Kepler's third law of *n*-body system

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Newton's gravitational law and its derived Kapler's orbit of planetary motions are the very first triumph of science, which confirms that a purely human's invention of mathematics is able to accurately describe observations in nature. For the past three centuries, the frontier of science has moved to almost all areas of human experience, yet some basic problem in celestial mechanics has remained unsolved, despite the effort of many great minds like Newton (1687), Euler (1760) and Lagrange (1776), among others.

The famous three-body problem, originating from the Sun-Earth-Moon system under the Newtonian gravitation field, is particularly attractive, for which Oscar II, King of Sweden, established a prize in 1887 for anyone who could find the solution. The prize was finally awarded to Henri Poincaré, who first found that there is no general analytical solution [1]. It is now recognized that the ensemble of its orbits has an enormous complexity. Recently, thanks to the advance of computational techniques, some breakthroughs begin to emerge: a figure-eight orbit was discovered numerically by Moore in 1993 [2], and 13 new distinct periodic orbits were found by Šuvakov and Dmitrašinović [3] in 2013, when three masses are equal with zero angular momentum. In 2017, Li and Liao [4], Li et al. [5] reported a further major advance: 695 periodic orbits with equal masses [4], and 1223 periodic orbits found later with unequal masses [5].

These findings have led to a fundamental conjecture that the 3-body system (m_1, m_2, m_3) may obey a kind of law of "harmonies": in analogy to the Kepler's third law of the two-body problem, Šuvakov and Dmitrašinović [6] proposed that a generalized Kepler's third law could be in the form $T|E|^{3/2}$ = constant, where $|E|$ denotes the sum of kinetic and potential energy of the 3-body system, while T is the period of periodic orbit, to be referred to as the harmonics of the system.

The question then remains whether $T|E|^{3/2}$ = constant is universal, and if not, what form it will takes. More interestingly, does there exist similar relation for an n-body system? This is a question that can not be answered by computation alone, especially for $n > 3$ in the presence of an even greater complexity than 3-body problem. In this case, an analytic approach, if successful, would be likely valuable.

Prof. Bohua Sun from the Cape Peninsula University of Technology, South Africa, obtained an epic result from the perspective of dimensional analysis [7], by working with a reduced gravitation parameter α_n, then predicting a dimensionless relation $T_n|E_n|^{3/2} = \text{const} \times \alpha_n\sqrt{\mu_n}$ (μ_n is reduced mass). The const= $\frac{\pi}{\sqrt{2}}$ is derived by matching with the 2-body Kepler's third law, and then a surprisingly simple relation for Kepler's third law of an n-body system is derived by invoking a symmetry constraint inspired from Newton's gravitational law:

$$T_n|E_n|^{3/2} = \frac{\pi}{\sqrt{2}} G\left[\frac{\sum_{i=1}^{n}\sum_{j=i+1}^{n}(m_i m_j)^3}{\sum_{k=1}^{n} m_k}\right]^{1/2}, \tag{1}$$

where the point masses, $m_1, m_2, ..., m_N$, the gravitational constant, G, the orbit period, T_n, and the system total energy, $|E_n|$. This formulae is, of course, consistent with the Kepler's third law of 2-body system, but yields a non-trivial prediction of

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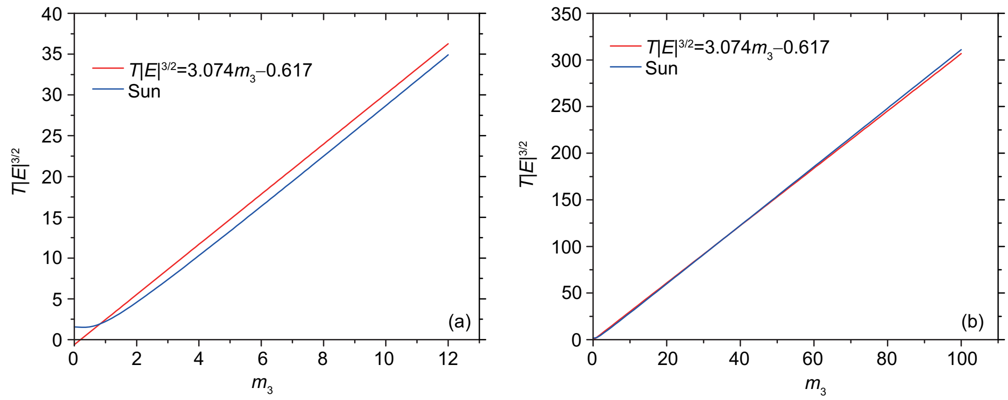

Figure 1 (Color online) Comparing for different m_3, (a) is for $m_3 \in [0, 12]$, and (b) is for $m_3 \in [0, 100]$. Red line is from ref. [5].

the Kepler's third law of 3-body:

$$T_3|E_3|^{3/2} = \frac{\pi}{\sqrt{2}} G \left[\frac{(m_1 m_2)^3 + (m_1 m_3)^3 + (m_2 m_3)^3}{m_1 + m_2 + m_3} \right]^{1/2}. \quad (2)$$

These analytic relations, eq. (1) and eq. (2), are obtained for the first time, and so present a significant step towards quantitative description of 3-body and n-body system. Note that the expression on the right hand side of eq. (1) or eq. (2) involves the sum of all masses in the denominator, and the terms proportional to the product of two masses only in the numerator; this construction is a guess at the present stage, possibly involving a deeper symmetry content worthy further exploration. Fortunately, numerical results of Li et al. [5] are available for a verification: for 3-body system with $G = 1$ and $m_1 = m_2 = 1$ and varying m_3, the predicted result, $T_3|E_3|^{3/2} = \frac{\pi}{\sqrt{2}} \left[\frac{1+2(m_3)^3}{2+m_3} \right]^{1/2}$, is in close agreement with numerical finding of ref. [5]: $(T_3|E_3|^{3/2})_{\mathrm{NUM}} \approx 3.074 m_3 - 0.617$, as shown in Figure 1.

Two specific features are noteworthy. First, in the limit of $m_3 \gg 1$, eq. (2) predicts that $T_3|E_3|^{3/2} \approx \pi(m_3 - 1)$, a linear relation on m_3 providing an explanation for the numerical slope 3.074. Furthermore, eq. (2) predicts that $T_3|E_3|^{3/2} \approx \pi/2 > 0$ in the limit of $m_3 \to 0$, improving the numerical formulae at that limit, which becomes negative and so inapplicable.

Note that the small but finite difference between eq. (2) and the numerical results are very interesting, since eq. (2) is an analytic construction extended from the exact 2-body Kepler's third law under a symmetry argument. Is the small difference a signature of the breaking-down of the symmetry (from 2-body to 3-body), or the numerical solutions still present some systematic bias (e.g. for increasing m_s)? This is intriguing to investigate in the future.

Back to the classical 3-body problem, eq. (2) partially answers the conjecture proposed by ref. [6]. Note that the periodic orbits form a subset of all solutions, and it is likely that eq. (2) captures only a subset of all periodic orbits. Nevertheless, if proved to be correct, one may capture a very important subset of all solutions. Furthermore, from a practical point of view, celestial planetary system is typically n-body system, eq. (1) would provide a helpful guide in the search for new periodic orbit of a practical n-body system whose numerical study is generally very resource-demanding. Whether eq. (1) is confirmed or disproved, important physics will be learnt. It is particularly interesting to test eq. (1) in practical celestial system like our planetary system. Finally, even if eq. (1) is confirmed numerically, rigorous mathematical proof is desired, since the heuristic argument behind eq. (1) needs to find its limit of applicability. We thus conclude that [7] opens a new avenue for further study of the classical n-body problem of mechanics.

1 H. Poincaré, Acta Math. **13**, 5 (1890).
2 C. Moore, Phys. Rev. Lett. **70**, 3675 (1993).
3 M. Šuvakov, and V. Dmitrašinović, Phys. Rev. Lett. **110**, 114301 (2013), arXiv: 1303.0181.
4 X. M. Li, and S. J. Liao, Sci. China-Phys. Mech. Astron. **60**, 129511 (2017), arXiv: 1705.00527.
5 X. M. Li, Y. P. Jing, and S. J. Liao, arxiv.org/abs/1709.04775.
6 V. Dmitrašinović, and M. Šuvakov, Phys. Lett. A **379**, 1939 (2015), arXiv: 1507.08096.
7 B. H. Sun, Sci. China-Phys. Mech. Astron. **61**, 054721 (2018).