Pergamon

Int. J. Solids Structures Vol. 33, No. 16, pp. 2299–2308, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00153–0

# EXACT DISPLACEMENT SOLUTION OF ARBITRARY DEGREE PARABOLOIDAL SHALLOW SHELL OF REVOLUTION MADE OF LINEAR ELASTIC MATERIALS

BO-HUA SUN
Institute of Technology, Peninsula Technikon, P.O. Box 1906, Bellville 7530, South Africa

WEI ZHANG
Department of Engineering Mechanics, Tsinghua University, Beijing 100084, People's Republic of China

KAI-YUAN YEH
Department of Mechanics, Lanzhou University, Gansu 73000, People's Republic of China

and

F. P. J. RIMROTT
Department of Mechanical Engineering, University of Toronto, Ontario, Canada

(*Received* 12 *January* 1995; *in revised form* 7 *July* 1995)

**Abstract**—The original contribution in the present paper can be described as: using the equilibrium equations of paraboloidal shallow shells expressed by displacements and by introducing a displacement function $U$, the general solution of the equation can be represented by $U(r, \varphi)$, and the equations can be reduced to single eighth-order equation of $U(r, \varphi)$. The displacement function introduced initially by the authors plays an important role as does Vlasov's displacement function for the cylindrical shell. The exact series form solution of the equation has been obtained for the bending problem of the shells. Numerical examples have been carried out for the shell with clamped boundary condition. Copyright © 1996 Elsevier Science Ltd.

## NOTATIONS

| | |
|---|---|
| $U(r, \varphi)$ | displacement function |
| $u, v, w$ | components of displacement |
| $r, \varphi, z$ | coordinates |
| $m$ | shape parameter of the shells |
| $f$ | arch height |
| $R$ | maximum radius of shell |
| $x = r/R$ | dimensionless radius variable |
| $R_i (i = 1, 2)$ | radius of curvature |
| $\kappa_i (i = 1, 2)$ | curvatures |
| $A_i (i = 1, 2)$ | Lamé parameters |
| $\varepsilon_1, \varepsilon_2, \omega$ | strains |
| $\chi_1, \chi_2, \tau$ | change of curvatures |
| $T_1, T_2, S$ | components of forces of stress resultants |
| $M_1, M_2, H$ | components of couples of stress resultants |
| $K = Eh/(1-\mu^2)$ | stiffness in the mid-surface of the shells |
| $D = Eh^3/12(1-\mu^2)$ | bending stiffness |
| $E$ | Young's modulus |
| $\mu$ | Poisson's ratio |
| $h$ | thickness of the shells |
| $q_1, q_2$ and $q_n$ | components of loads |
| $L_{ij}$ | differential operators |
| $D_x(:)$ | $= x\,\partial(:)/\partial x$ |
| $\delta_x(:)$ | $= x\,\mathrm{d}(:)/\mathrm{d}x$ |
| $\partial_{,\varphi}(:)$ | $= \partial(:)/\partial\varphi$ |
| $U_i$ | particular solutions |
| $Z_v(\ldots)$ | solution of $v$-order Bessel equation |
| ${}_pE_q(\ldots)$ | first generalized hypergeometric function |
| ${}_p\Phi_q(\ldots)$ | second generalized hypergeometric function |

## 1. INTRODUCTION

The calculation of a shallow paraboloidal shell of revolution with arbitrary degree of parabola meridian is often met in pressure vessel design and aerospace engineering. Due to the difficulty of the problem, numerical analysis, e.g. finite element analysis (FEA), has to be used in practical designs because of its versatility. We believe that analytical solutions are always important, however, even in the case of having a powerful numerical method both from an academic and an engineering point of view; the analytical solution is very accurate and straightforward, and it provides closed-form solutions that permit analytical study of the effects of changing various parameters. Conversely, a specific numerical result can only be found for a specific problem with special parameters (Cook *et al.*, 1989). Obviously, the analytical analysis also has the disadvantage that it can only be obtained for simple loading, geometry and boundary conditions. Even for simple problems, the work of finding an analytical solution is always difficult and challenging, and that is the reason why people prefer finite element solutions. It does not mean that closed-form solutions have no practical application.

As we know, analytical solutions have been obtained for some shells with simpler geometrical shapes, e.g. cylindrical shells, conical shells, and spherical shells. Until now, no favoured displacement solution has been obtained for the shallow shell of revolution with an arbitrary degree of parabola meridian. In this paper we shall shoulder this heavy responsibility of finding an analytical solution of the problem.

There is little literature to which one can refer. As pioneers, Luo and Pan (1967) were the first to derive a simplified complex form differential equation and gave the homogeneous solution for the *axisymmetric problem* of the shells using Thomson's function under Gekerler's simplification. The exact general solution for the complex form equation of the shells, which was established by Luo and Pan (1967), was first obtained by Sun (1989). However, the solutions reached by Luo and Pan (1967) and Sun (1989) are not easy to apply to structures composed of shells. On the other hand, the equation derived is in the complex form and cannot easily be used to treat dynamical problems.

There is clearly a need for an equation which can be used not only for bending and buckling but also for vibration problems of the shells. It is clear that such equations must be given in terms of displacement components. Parts of Sun (1989, 1991a, 1991b) focused attention on this subject and are the basis of this paper.

The goal of the present paper is to set up the equilibrium equations in terms of displacements for arbitrary degree paraboloidal shallow shell of revolution made of linear elastic material subjected to an arbitrary static loading, and to give its exact analytical solution by using the Bessel functions and hypergeometrical ones. To the authors' knowledge, this kind of problem has never been studied before, and all the results of the present paper are original. The displacement function of the shell introduced by the authors plays an important role, as does Vlasov's displacement function for cylindrical shells.

To make the paper self-contained, it is important to repeat a part of the results obtained previously. In Section 2, the geometrical properties of the shells and the approximations of principal curvatures or Lamé parameters, due to the assumption that the shell is shallow, are illustrated, and the governing equations in terms of displacement components are given for a linear elastic material under introduction of a special partial differential operator. The characteristics of these governing equations are discussed before solving the problems.

In Section 3, using the property of the differential operator $D_x(:) = x\,\partial(:)/\partial x$ under the variable transformation $x = \exp(t)$, where $t$ is a new independent variable, the general solution of the governing equations is found and the equations are transformed into a single equation with an unknown $U(x, \varphi)$ which is called a displacement function for the shells in this paper.

In Section 4, the solutions of the axisymmetric bending problem of the shells are presented in the Bessel functions. In Section 5, the solutions of asymmetric problems have been given in the hyper-geometric functions for the first time.

Finally, a numerical example for axisymmetric bending of the shell with clamped boundary under distribution loading has been carried out in detail. Through numerical comparison for uniform normal load, it is revealed that the paraboloidal shell of second

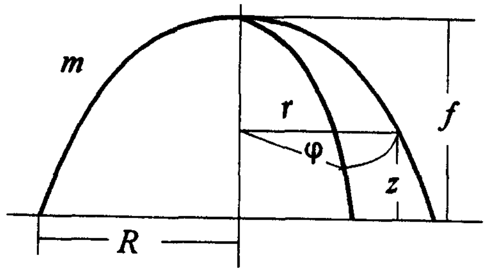


Fig. 1. Geometry of shells.

degree, i.e. the shallow spherical shell, is the most favourable design under this particular loading. The optimisation of shape parameter $m$ can also be studied using the solutions obtained in this paper.

It is worth noting that the displacement function introduced here can also be used to treat the buckling and vibration problem of the shell. These are to be considered in forthcoming papers.

## 2. FORMULATIONS

The formulations of the problem are based on the shell theory of Novozhilov (1970). The notations and meaning of quantities, especially the assumption of shallow shell, can be found in the book by Novozhilov (1970).

### 2.1. *Geometrical description and the approximations for paraboloidal shallow shells*

The middle surface of a paraboloidal shallow shell is given by the following equations (Luo and Pan, 1967; Sun, 1989)

$$\alpha_1 = r, \quad \alpha_2 = \varphi, \quad z = f(1-x^m) \tag{1}$$

where $f$ is the arch height of middle surface, $R$ is the maximum radius, $x = r/R$ is a dimensionless radius variable, $m$ is an arbitrary exponent $m \geqslant 1$ (for conical shell, $m = 1$ and for spherical shell, $m = 2, \ldots$). The geometry of the shell is illustrated in Fig. 1.

According to the assumption of shallow shell given by Novizhilov (1970), the two approximate principal curvatures and Lamé parameters for shallow shells of revolution are defined (Luo and Pan, 1967; Novizhilov, 1970; Sun, 1989), respectively:

$$k_1 = \frac{1}{R_1} \approx \frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = -\frac{m(m-1)f}{R^2}x^{m-2}$$

$$k_2 = \frac{1}{R_2} \approx \frac{1}{r}\frac{\mathrm{d}z}{\mathrm{d}r} = -\frac{mf}{R^2}x^{m-2}, \quad A_1 = 1, \quad A_2 = r. \tag{2}$$

### 2.2. *Strains and change of curvatures of the middle surface in terms of the displacement components* u, v, w

For the shallow paraboloidal shells of revolution, we have

$$\varepsilon_1 = u_{,r} + \frac{w}{R_1}, \quad \varepsilon_2 = \frac{u}{r} + \frac{v_{,\varphi}}{r} + \frac{w}{R_2}, \quad \omega = r\left(\frac{v}{r}\right)_{,r} + \frac{u_{,\varphi}}{r}$$

$$\chi_1 = -w_{,rr}, \quad \chi_2 = -\frac{w_{,\varphi\varphi}}{r^2} - \frac{w_{,r}}{r}, \quad \tau = -\frac{w_{,r\varphi}}{r} + \frac{w_{,\varphi}}{r^2} \tag{3}$$

where $\varepsilon_1$, $\varepsilon_2$, $\omega$ are strains; $\chi_1$, $\chi_2$, $\tau$ are change of curvatures.

### 2.3. *Constitutive equations*

For shells made of linear elastic materials, we have

$$T_1 = K(\varepsilon_1+\mu\varepsilon_2),\quad T_2 = K(\varepsilon_2+\mu\varepsilon_1),\quad S=\tfrac{1}{2}K(1-\mu)\omega$$
$$M_1 = D(\chi_1+\mu\chi_2),\quad M_2 = D(\chi_2+\mu\chi_1),\quad H = D(1-\mu)\tau \tag{4}$$

in which $T_1$, $T_2$, $S$ denote the components of forces of stress resultants, $M_1$, $M_2$, $H$ denote the components of couples of stress resultants, and $K = Eh/(1-\mu^2)$, $D = Eh^3/12(1-\mu^2)$. $E$ is Young' modulus, $\mu$ is Poisson' ratio, and $h$ is the thickness of the shell. The quantities used here can be found in Novozhilov (1970).

### 2.4. *The equilibrium equations in terms of displacements*

Substituting eqn (3) into eqn (4), and then into equilibrium equations of shells (Novozhilov, 1970), we have the following equilibrium equations in terms of displacement components

$$L_{11}(u)+L_{12}(v)+L_{13}(W)+\frac{q_1 r^2}{K}=0$$
$$L_{21}(u)+L_{22}(v)+L_{23}(W)+\frac{q_2 r^2}{K}=0$$
$$L_{31}(u)+L_{32}(v)+L_{33}(W)=\frac{q_n r^4}{D} \tag{5}$$

where

$$W = mfx^{m-1}\frac{w}{R},\quad D_x(:) = x\frac{\partial(:)}{\partial x},\quad \partial_{,\varphi}(:) = \frac{\partial(:)}{\partial\varphi}$$
$$L_{11} = D_x^2-1+\frac{1-\mu}{2}\partial_{,\varphi}^2,\quad L_{12} = \left(\frac{1+\mu}{2}D_x-\frac{3-\mu}{2}\right)\partial_{,\varphi},\quad L_{13} = aD_x-b$$
$$L_{21} = \left(\frac{1+\mu}{2}D_x+\frac{3-\mu}{2}\right)\partial_{,\varphi},\quad L_{22} = \frac{1-\mu}{2}(D_x^2-1)+\partial_{,\varphi}^2,\quad L_{23} = b\,\partial_{,\varphi}$$
$$L_{31} = \frac{mfRK}{D}x^{m+1}(aD_x+b),\quad L_{32} = bmfR\frac{K}{D}x^{m+1}$$
$$L_{33} = [(D_x-2)^2+\partial_{,\varphi}^2](D_x^2+\partial_{,\varphi}^2)+cm^2f^2\frac{K}{D}x^{2m}$$
$$a = m-1+\mu,\quad b = (m-1)\mu+1,\quad c = (m-1)(m-1+2\mu)+1 \tag{6}$$

$q_1$, $q_2$, and $q_n$ denote the components of the load applied on the shell. The mathematical problem of the shell is to solve eqn (5) under boundary conditions. Unfortunately, the solution of eqn (5) is very problematic due to the variability of its coefficients. In the following section, we will reduce eqn (5) into a single equation by introducing a displacement function, which is possible because of the properties of the dimensionless Euler operator $D_x(:)$, which might be considered as a generalised Euler operator.

## 3. THE DISPLACEMENT FUNCTION AND THE GENERAL SOLUTION OF BENDING PROBLEM

Let us make a coordinate transformation, $x = \exp(t) = e^t$. Under the transformation $x = \exp(t)$, the operator $D_x(:)$ can be changed into the operator $\partial(:)/\partial t$, i.e. the operators

$L_{ij}$ ($i = 1, 2$; $j = 1, 2, 3$) will be changed into operators with constant coefficients. The general solution of eqn (5) can be represented in a displacement function $U(x, \varphi)$, i.e.

$$\begin{aligned} u &= L_u(U) - L_{22}(U_1) + L_{12}(U_2) \\ v &= L_v(U) + L_{21}(U_1) - L_{11}(U_2) \\ w &= \frac{1-\mu}{2}\frac{R}{mf}x^{1-m}L_w(U) \end{aligned} \tag{7}$$

where

$$\begin{aligned} L_u &= b\left(\frac{1+\mu}{2}D_x - \frac{3-\mu}{2}\right)\partial^2_{,\varphi} - (aD_x - b)\left[\frac{1-\mu}{2}(D_x^2 - 1) + \partial^2_{,\varphi}\right] \\ L_v &= -b\left(D_x^2 - 1 + \frac{1-\mu}{2}\partial^2_{,\varphi}\right)\partial_{,\varphi} + (aD_x - b)\left(\frac{1+\mu}{2}D_x + \frac{3-\mu}{2}\right)\partial_{,\varphi} \\ L_w &= [D_x^2 + \partial^2_{,\varphi}]^2 - 2[D_x^2 - \partial^2_{,\varphi}] + 1 \end{aligned} \tag{8}$$

and $U_i (i = 1, 2)$ are the particular solutions of the following equations

$$[(D_x^2 + \partial^2_{,\varphi})^2 - 2(D_x^2 - \partial^2_{,\varphi}) + 1]U_i = \frac{2R^2x^2q_i}{[(1-\mu)K]} \quad (i = 1, 2). \tag{9}$$

It can be proved that the first two parts of eqn (5) are automatically satisfied by the general solution (7). Substitute eqn (7) into the third of eqn (5) and we have an equation of the displacement function $U(x, \varphi)$ as follows

$$L_1L_2(U) + m^2f^2\frac{K}{D}x^{2m}L_3(U) = 2\frac{P_q(x, \varphi)}{1-\mu} \tag{10}$$

in which

$$\begin{aligned} L_1 &= [(D_x - m - 1)^2 + \partial^2_{,\varphi}][(D_x - m + 1)^2 + \partial^2_{,\varphi}], \\ L_2 &= L_w, \quad L_3 = cL_w + \frac{2}{1-\mu}[(aD_x + b) + bL_v], \end{aligned}$$

$$P_q(x, \varphi) = mfR^3x^{m+3}\frac{q_n}{D} + m^2f^2\frac{K}{D}x^{2m}[(aD_x + b)L_{22} - bL_{21}]U_1 + [bL_{11} - (aD_x + b)L_{12}]U_2. \tag{11}$$

The general solution and basic equation of axisymmetric deformation are as follows

$$u = -(a\delta_x - b)U - U_1, \quad w = \frac{R}{mf}x^{1-m}[\delta_x^2 - 1]U \tag{12a}$$

$$(\delta_x - m - 1)^2(\delta_x - m + 1)^2(\delta_x^2 - 1)U + m^2f^2(1-\mu^2)\frac{K}{D}x^{2m}[\delta_x^2 - (m-1)^2]U = P_q(x) \tag{12b}$$

$$(\delta_x^2 - 1)v = R^2x^2\frac{q_2}{K} \tag{12c}$$

where

$$\delta_x = x\frac{\mathrm{d}(:)}{\mathrm{d}x} \quad P_q(x) = mfR^3x^{m+3}\frac{q_n}{D}+m^2f^2\frac{K}{D}x^{2m}(a\delta_x+1)U_1. \tag{13}$$

Equations (12a,b) denote the axisymmetric bending problem. Equation (12c) denotes the pure torsion problem of the shell. It is shown that the shearing stress distribution due to torsion is independent of other stress components including those of membrane stress and bending stress, if $q_2 = 0$ and $u_2 = 0$.

It is worth noting that the operator $D_x(:) = x\,\partial(:)/\partial x$ plays an important role in introducing the displacement function $U$, which is generally called dimensionless Euler operator. It was originally introduced by Sun (1986) and very successfully applied in Sun (1987, 1989, 1991a,b), Sun and Huang (1989), Huang and Sun (1986).

Clearly, the contact forces and couples of stress resultants can also be represented in the displacement function $U(x, \varphi)$ by substituting eqn (7) into eqn (3) and then into eqn (4).

## 4. THE AXISYMMETRIC BENDING PROBLEM OF PARABOLOID SHALLOW SHELLS

The eqn (12b) can be rewritten as follows

$$[(\delta_x-m)^2-1+\mathrm{i}\lambda_0x^m](\delta_x^2-1-\mathrm{i}\lambda_0x^m)Y = T(x) \tag{14}$$

where

$$\lambda_0 = mf\sqrt{(1-\mu^2)\frac{K}{D}} \quad U = x^{m-1}\int Yx^{-m}\,\mathrm{d}x+c_2x^{m-1}$$

$$T(x) = x^{m+1}\int x^{-(m+1)}P_q(x)\,\mathrm{d}x+c_1x^{m+1}. \tag{15}$$

The homogeneous solution of eqn (14) can be written as

$$Y = (c_3+\mathrm{i}c_4)x^\alpha Z_{\mu_1}(\lambda_1x^{\beta_1})+(c_5+\mathrm{i}c_6)Z_{\mu_2}(\lambda_2x^{\beta_2}) \tag{16}$$

in which $Z_v(\ldots)$ is the solution of $v$-order Bessel equation (Wang and Guo, 1979), and the parameters are

$$\alpha = \frac{m+1}{2}, \quad \mu_1 = \frac{2\sqrt{m+5/2}}{m}, \quad \lambda_1 = \frac{2\sqrt{\mathrm{i}\lambda_0}}{m}$$

$$\beta_1 = \frac{m}{2}, \quad \mu_2 = \frac{2}{m}, \quad \lambda_2 = \frac{2\sqrt{-\mathrm{i}\lambda_0}}{m}, \quad \beta_2 = \frac{m}{2} \tag{17}$$

and $c_1$ and $c_2$ are constants to be determined by boundary conditions of given problems.

## 5. ASYMMETRIC BENDING PROBLEM OF PARABOLOIDAL SHALLOW SHELLS

In this case, any unknown functions and loads may be expanded in Fourier serial form as follows

$$(:) = \sum_{n=0}^{\infty}(:)_{ns}\sin n\varphi+\sum_{n=0}^{\infty}(:)_{nc}\cos n\varphi. \tag{18}$$

Equation (10) becomes

$$B_1(U_{nc})+m^2f^2\frac{K}{D}x^{2m}[B_2(U_{nc})+B_3(U_{ns})-B_4(U_{ns})] = P_{nc}$$

$$B_1(U_{ns})+m^2f^2\frac{K}{D}x^{2m}[B_2(U_{ns})+B_3(U_{nc})-B_4(U_{nc})] = P_{ns} \tag{19}$$

in which

$$B_1 = [(\delta_x-m-1)^2-n^2][(\delta_x-m+1)^2-n^2][(\delta_x^2-n^2)^2-2(\delta_x^2+n^2)+1]$$

$$B_2 = c[(\delta_x^2-n^2)^2-2(\delta_x^2+n^2)+1]$$

$$B_3 = \frac{2}{1-\mu}(a\delta_x+b)\left[n^2b\left(\frac{1+\mu}{2}\delta_x-\frac{3-\mu}{2}\right)-(a\delta_x-b)\left(\frac{1+\mu}{2}(\delta_x^2-1)-n^2\right)\right] \tag{20a}$$

$$B_4 = \frac{2n}{1-\mu}\left[-b\left(\delta_x^2-1-\frac{1+\mu}{2}n^2\right)+(a\delta_x-b)\left(\frac{1+\mu}{2}\delta_x+\frac{3-\mu}{2}\right)\right]$$

$$P_{nc} = mfR^3x^{m+3}\frac{q_{nnc}}{D}+m^2f^2\frac{K}{D}x^{2m}\left[(a\delta_x+b)\left(\frac{1-\mu}{2}(\delta_x^2-1)-n^2\right)U_{1nc}\right.$$

$$-nb\left(\frac{1+\mu}{2}\delta_x+\frac{3-\mu}{2}\right)U_{1nc}+b\left(\delta_x^2-1-\frac{1-\mu}{2}n^2\right)U_{2nc}$$

$$\left.-n(a\delta_x+b)\left(\frac{1+\mu}{2}\delta_x-\frac{3-\mu}{2}\right)U_{2ns}\right],\quad P_{ns} = P_{nc}[c\to s, n\to n]. \tag{20b}$$

In order to simplify eqn (19), let us introduce two complex functions $F = U_{nc}+\mathrm{i}U_{ns}$, $Q = P_{nc}+\mathrm{i}P_{ns}$, where $\mathrm{i} = \sqrt{-1}$. Equation (19) becomes

$$B_1(F)+m^2f^2\frac{K}{D}x^{2m}[a_1\delta_x^4+a_2\delta_x^2+a_3\delta_x+a_4]F = Q(x). \tag{21}$$

Equation (21) can be rewritten as follows

$$\prod_{i=1}^{8}\left(\delta_\zeta+\frac{\alpha_i}{2m}\right)F-\zeta\prod_{i=1}^{4}\left(\delta_\zeta+\frac{\beta_i}{2m}\right)F = \frac{Q(\zeta)}{(2m)^4} \tag{22}$$

in which

$$\zeta = -\frac{m^2f^2}{(2m)^4}\frac{K}{D}a_1x^{2m},\quad \alpha_1 = n+m+1,\quad \alpha_2 = -n+m+1,\quad \alpha_3 = n+1$$

$$\alpha_4 = -n+1,\quad \alpha_5 = n+m-1,\quad \alpha_6 = -n+m-1,\quad \alpha_7 = n-1,\quad \alpha_8 = -n-1. \tag{23}$$

$\beta_i$ ($i = 1, 2, 3, 4$) are the roots of the following algebraic equation

$$\beta^4+\frac{a_2}{a_1}\beta^2+\frac{a_3}{a_1}\beta+\frac{a_4}{a_1} = 0 \tag{24}$$

in which the parameters are

$$a_1 = c-\frac{1+\mu}{1-\mu}a^2,$$

$$a_2 = -2c(n^2+1)+\frac{1+\mu}{1-\mu}(a^2+b^2-abn^2+\mathrm{i}an)+\frac{2n}{1-\mu}(a^2n-\mathrm{i}b)$$

$$a_3 = \frac{3-\mu}{1-\mu}an(bn+\mathrm{i}) - \frac{1+\mu}{1-\mu}bn(b+\mathrm{i}),$$

$$a_4 = c(n^2-1)^2 + \frac{3-\mu}{1-\mu}bn(bn-\mathrm{i}) - \frac{2n}{1-\mu}(b^3n^2-\mathrm{i}) + \mathrm{i}n^3\frac{1+\mu}{1-\mu}. \quad (25)$$

Equation (22) is a Meijer equation. It can be changed into a generalized hypergeometric equation by proper variable transformation; we have

$$\left[\delta_\zeta \prod_{i=1}^{7}(\delta_\zeta+\omega_i-1) - \zeta\prod_{j=1}^{4}(\delta_\zeta+\lambda_j)\right]\Psi = \frac{Q(\zeta)\zeta^{\alpha_1/(2m)}}{16m^4} \quad (26)$$

where

$$\omega_i = 1-\frac{\alpha_1-\alpha_{i+1}}{2m}, \quad \lambda_j = \frac{\beta_j-\alpha_1}{2m}, \quad \Psi = \zeta^{\alpha_1/(2m)}F. \quad (27)$$

The homogeneous solution of eqn (26) can be represented by a hypergeometric function

$$\Psi^h = \sum_{e=0}^{3} C_e\zeta^{1-\omega_e}{}_4F_7(a_{1e}, a_{2e}, a_{3e}, a_{4e}; b_{0e}, b_{1e}^*, \ldots, b_{7e}; \zeta)$$

$$+ \sum_{e=0}^{3} C_e^*\zeta^{1-\omega_e}{}_4\Phi_7(a_{1e}, \ldots, a_{4e}; b_{0e}, b_{1e}^*, \ldots, b_{7e}; \zeta) \quad (28)$$

in which $C_e$ and $C_e^*$ are arbitrary boundary constants. ${}_pF_q(\ldots)$ and ${}_p\Phi_q(\ldots)$ are the first and second generalized hypergeometric function (Kovalenko, 1963), respectively.

In the case of distributed load acted on the shell, $Q(\zeta)$ can be written as follows

$$Q(\zeta) = \sum_{v=0}^{N} Q_v\zeta^{\sigma_v} \quad (30)$$

in which $Q_v$ are known constants, and $N$ is an arbitrary integer. For each term $Q_v\zeta^{\sigma_v}$ of eqn (30), eqn (26) becomes

$$\left[\delta_\zeta \prod_{i=1}^{7}(\delta_\zeta+\omega_i-1) - \zeta\prod_{j=1}^{4}(\delta_\zeta+\lambda_j)\right]\Psi_v^p = \frac{Q_v\zeta^{\sigma_v+\alpha_1/2m}}{16m^4}. \quad (31)$$

We have the particular solution of the problem

$$\Psi^p = \sum_{v=0}^{N}\Psi_v^p = \sum_{v=0}^{N}\frac{Q_v\zeta^{\sigma_v+\frac{\alpha_1}{2m}}}{16m^4\prod_{i=1}^{8}\left[\sigma_v+\frac{\alpha_i}{2m}\right]}{}_5F_8(\mu_{v1}, \ldots, \mu_{v4}; \gamma_{v1}, \ldots, \gamma_{v8}; \zeta)$$

$$\mu_{vi} = \frac{\beta_i}{2m}+\sigma_v, \quad \gamma_{vi} = 1+\sigma_v+\frac{\alpha_i}{2m}. \quad (32)$$

It is worth noting that eqn (30) is not a Taylor's series of $Q(\zeta)$, but is only the general representation of distributed loads.

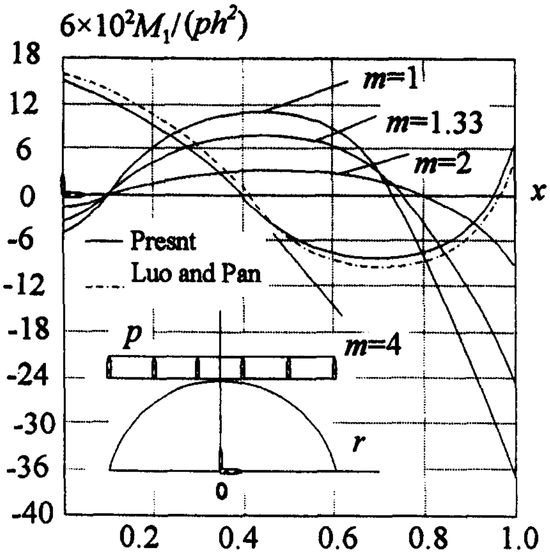


Fig. 2. Couples in the shell for $m$ = 1, 1.33, 2, 4.

## 6. NUMERICAL EXAMPLE AND CONCLUSIONS

We consider a shell of constant thickness. The lower edge is clamped. It is assumed that Young's modulus $E = 10^7$, Poisson's ratio is 0.3, and $h/R = 0.01$, $f/R = 0.05$. The applied load is a surface load that is uniformly distributed over the middle surface (here positive outward). We wish to determine the couples in the shell due to this load. By computing, we obtain the values plotted graphically in Fig. 2. It will be seen that the values of the couples in the central part of the shell agree closely with the values furnished by the membrane theory, while edge effects are significant in the vicinity of the edges. This kind of behaviour is typical of thin shell structures.

We believe that the following conclusions can be reached:

(1) The analytical solution can be obtained by introducing an intelligent variable transformation.
(2) Our formulations of the shell, eqns (7), (9) and (10), are quite general; for instance, the displacement function introduced in this paper is also valid for the vibration and buckling of the shell.
(3) The solutions obtained here are useful to optimise the shape of the shell because, in this case, we have only one shape parameter $m$ concerned for different loading conditions. For uniform load, it reveals that the paraboloidal shell of second degree, i.e. the shallow spherical shell, is a favourable design for this paticular loading.
(4) The approach here can also be generalised to treat sandwich paraboloidal shells with special relations between cores and layers.
(5) Our basic eqn (10) has a dramatic form which is needed for an asymptotic expansion solution (Goldenveizer, 1961; Steele, 1989), which will be given in forthcoming papers.

*Acknowledgements*—The authors wish to express sincere thanks to the support of China National Postdoctoral Foundation (CNPF) (July 1989–July 1991), comments from reviewers and Ms Jean Irving of University of Cape Town for her kind help in editing the manuscript.

## REFERENCES

Cook, D. R., David, S. M. and Michael, E. P. (1989). *Concepts and Applications of Finite Element Analysis* (3rd edn). John Wiley.

Goldenveizer, A. L. (1961). *Theory of Elastic Shells*, translated from Russian by G. Herrmann. Pergamon Press, New York, 658pp.

Huang, Y. and Sun, B.-H. (1986). The general solution of conical shells on the elastic foundation. *Shells, Membrances and Space Frames, Proc. IASS Symp.* Osaka, Vol. 1, pp. 193–200.

Kovalenko, A. D. (1963). *The Theory of Circular Conical Shells and Its Application in Mechanical Manufacture*. Academy of Science of Ukrainian SSR, Kiev.

Lou, T. and Pan, C. (1967). Axisymmetric bending of paraboloidal shallow shells. *Acta Mech. Sin.* **6,** 200–218.

Novozhilov, V. V. (1970). *Theory of Thin Shells*, translated from Russian by P. G. Lowe. Wolters-Noordhoff, Groningen, 422pp.

Steele, C. R. (1989). Asymptotic analysis and computation for shells. In *Analytical and Computational Models of Shells*, Vol. 3. CED, New York.

Sun, B.-H. (1986). The united displacement theory of conical shells. M.Sc.(Eng) Dissertation, Xian Institute of Metallurgy and Construction Engineering, China.

Sun, B.-H. (1987). Exact solution of eigenfrequencies of conical shells. *Acta Mech. Sin.* **19**(2), 136–145.

Sun, B.-H. (1989). The general bending problem of paraboloidal shallow shells. *Acta Mech. Sin.* **21,** 245–248.

Sun, B.-H. (1991a). Some developments of applied mechanics. Post-Doctoral Report, Department of Engineering Mechanics, Tsinghua University, Beijing, pp. 1–126.

Sun, B.-H. (1991b). The displacement solution for the bending of paraboloidal shallow shells. *Proc. of Int. Conf. of APCS*-91, Beijing, Vol. 2, pp. 1–12.

Sun, B.-H. and Huang, Y. (1989). New development in classical theory of conical shells. *Adv. Mech.* **19**(4), 497–506.

Wang, Z. and Guo, D. (1979). *An Introduction to Specials Functions*. Science Press, Beijing.