Appl. Math. Mech. -Engl. Ed., **38**(6), 765–778 (2017)
DOI 10.1007/s10483-017-2204-8

Applied Mathematics and Mechanics (English Edition)

Scaling laws of compressible turbulence*

Bohua SUN†

Department of Mechanical Engineering, Cape Peninsula University of Technology, Cape Town 7535, South Africa

Abstract Spatial scaling laws of velocity kinetic energy spectra for the compressible turbulence flow and the density-weighted counterparts are formulated in terms of the wavenumber, dissipation rate, and Mach number by using a dimensional analysis. We apply the Barenblatt's incomplete similarity theory to both kinetic and density-weighted energy spectra. It shows that, within the initial subrange, both energy spectra approach the –5/3 and –2 power laws of the wavenumber when the Mach number tends to unity and infinity, respectively.

Key words compressible turbulence, spatial scaling law, dimensional analysis, incomplete similarity

Chinese Library Classification O303, O354, O357.5, O302
2010 Mathematics Subject Classification 76F02, 76F05, 76F50

Nomenclature

$\boldsymbol{u}$,	flow velocity;	C_ρ,	coefficient constant;
$\boldsymbol{v}$,	density-weighted velocity;	A,	adjustable parameter;
u'_i,	fluctuation of flow velocity;	σ,	adjustable parameter;
L,	length dimension;	β,	adjustable parameter;
t,	time dimension;	β_ρ,	adjustable parameter;
m,	mass dimension;	$E^{\rm c}$,	compressible part of E_ρ;
k,	wavenumber;	$E^{\rm s}$,	solenoidal part of E_ρ;
ν,	kinematic viscosity;	$c^{\rm c}$,	compressible part constant;
ε,	dissipation rate;	$c^{\rm s}$,	solenoidal part constant;
E,	kinetic energy spectrum;	λ,	constant;
E_ρ,	density-weighted energy spectrum;	$d(Ma)$,	exponent;
Ma,	Mach number;	$h(Ma)$,	exponent;
Re,	Reynolds number;	α,	exponent;
c,	speed of sound;	l,	box size;
γ,	ratio of specific heat;	l_ν,	box size of sequence ν;
η,	Kolmogorov length;	ρ,	mass density;
$C_{\rm K}$,	Kolmogorov constant;	ρ_0,	local reservoir value of density.
C,	coefficient constant;		

* Received May 30, 2016 / Revised Jan. 5, 2017
Project supported by the National Research Foundation of South Africa (No. 93918)
† Corresponding author, E-mail: sunb@cput.ac.za

1 Introduction

Turbulence is considered to be one of the unsolved problems in classical physics[1–8]. There are two kinds of turbulence, namely, incompressible and compressible. In recent years, compressible turbulence has drawn a great deal of attention. Fully developed three-dimensional homogenous incompressible turbulence has been studied by Kolmogorov[9–11], who showed that its energy spectrum exhibits $k^{-\frac{5}{3}}$ power of the wavenumber k in the inertial subrange. However, the basic processes, which occur in compressible turbulence, are less understood[1,12–22].

Kovasznay[23] pointed out that the compressible turbulence problem was characterized by the existence of acoustic, vortical and entropy modes, which are in interaction with each other. The systematic 2nd-approximation of these nonlinear interactions was shown by Chu and Kovasznay[17]. Lighthill[24] argued that its energy is continually radiated away in the form of sound waves, which are ultimately converted into heat by the various processes of acoustic attenuation. Therefore, one may visualize the compressibility effects as acting like a source of energy dissipation that is provided by viscosity and thermal conductivity[25]. As the nonlinear effects become prominent, the sound waves in compressible fluid sharpen to form shock waves, while the vortex formation behind the shock waves produces anisotropic shear turbulence. The investigation showed that the passage of a shock wave also results in smaller parallels to the shock and is compressed in the direction perpendicular to the shock. The shock formation and the shock interaction process lead to another source of energy dissipation in compressible turbulence.

Despite the anisotropy which is caused by the individual shock, Kadomtsev and Petviashvili[26] argued that the random orientation of various shocks leads to the overall isotropy of the turbulent field. Using a Burgers equation type model, they provided a spectrum for kinetic energy, $E(k) \sim \varepsilon c^{-1} k^{-2}$, where c is the speed of sound in fluid, k is the wavenumber, and ε is the dissipation rate. Moiseev et al.[25] applied group-invariance principles to the Hopf-type functional formulation of the compressible case, and gave the spectrum of the kinetic energy, $E(k) \sim \rho c^{\frac{2}{3\gamma-1}} \varepsilon^{\frac{2\gamma}{3\gamma-1}} k^{-\frac{5\gamma-1}{3\gamma-1}}$, where ρ is the density of fluid, and γ is the ratio of specific heats of fluid. Shivamoggi[27] asserted that Moiseev's spectrum is not completely correct and proposed a revised spectrum, i.e., $E(k) \sim \rho^{\frac{\gamma-1}{3\gamma-1}} c^{\frac{2}{3\gamma-1}} \varepsilon^{\frac{2\gamma}{3\gamma-1}} k^{-\frac{5\gamma-1}{3\gamma-1}}$.

However, it is easy to verify that the previous scaling laws[25–27] have a common problem, that is, the speed of the sound c and the density ρ do not appear in the dimensionless format. From our experience, they are supposed to be in the dimensionless form such as ρ_0/ρ or v/c. Therefore, the mentioned scaling laws of compressible turbulence might not be a proper format and should be investigated further.

High resolution numerical simulations for supersonic isothermal turbulence showed the $-\frac{5}{3}$ spectrum of both the velocity $\boldsymbol{u}$ and the density-weighted velocity $\boldsymbol{v} = \rho^{1/3}\boldsymbol{u}$ and proved that kinetic energy cascades conservatively in compressible turbulence, provided that the pressure-dilatation co-spectrum decays at a sufficiently rapid rate, which was supported once by numerical simulations[1,7,12–13,20]. Recently, for the Mach number Ma near 1, the numerical simulation from Ref. [7] confirmed that fully developed three-dimensional compressible turbulence density-weighted energy spectra exhibited Kolmogorov's $-\frac{5}{3}$ power law in the inertial range. Galtier and Banerjee[21] derived a relationship for the scaling of compressible isothermal turbulence, and showed that, only around the sonic scale, the density-weighted energy spectrum approached the $-\frac{5}{3}$ power law.

Since the numerical results can only be obtained for a specific case, it is very hard to predict the general scaling trends on the energy spectra by using limited numerical results, and it would be a natural attempt to find an alternative way. The new approach should be able to answer the following three questions: the first is how to predict scaling laws for an arbitrary Mach number, the second relates to whether the velocity kinetic energy $E(k, \varepsilon, Ma)$ can still be used to characterize the cascade of compressible turbulence, and the final question is in what form the scaling laws of the density-weighted energy spectrum $E_\rho(k, \varepsilon, Ma)$ for an arbitrary Mach number Ma will be.

Due to the complicated nature of compressible turbulence, the above questions may not be fully answered by numerical simulations, and hence alternative ways should be sought. Although the nature of compressible turbulence is still not fully understood, we believe that its physics must satisfy the dimensional laws. The results in this article show that the dimensional analysis can certainly capture the overall picture of the compressible turbulence cascade process, and leads to fairly rich information on the phenomena.

One of the most successful applications of dimensional analysis was formulated by Kolmogorov[9–11]. For homogenous incompressible turbulence, Kolmogorov introduced the length, time, and velocity scales of the smallest eddies of turbulence, whereas the smallest scales are given by the Kolmogorov's length $\eta = (\nu^3/\varepsilon)^{1/4}$, the energy-dissipation rate $\varepsilon = 2\nu\overline{s_{ij}s_{ij}}$, $s_{ij} = \frac{1}{2}(u'_{i,j} + u'_{j,i})^2$, and u'_i fluctuation of flow velocity. Kolmogorov stated that the large-scale turbulence motion is roughly independent of viscosity. The small scale, however, is controlled by viscosity. In the inertial range, turbulence is controlled solely by the dissipation rate ε and the size of eddy k. It is found that the spatial energy spectrum $E(k)$ can be formulated in terms of the wavenumber k and the dissipation rate ε as $E(k) = \varepsilon^{2/3}k^{-5/3}f((k\eta)^{4/3}) = C_{\mathrm{K}}\varepsilon^{2/3}k^{-5/3}$, which is the famous Kolmogorov's $-\frac{5}{3}$ power law of incompressible turbulence, where $f(\cdot)$ is a dimensionless function, and the universal constant $C_{\mathrm{K}} \approx 1.5$.

To make the paper self-contained, the paper is organized as follows. Following this introduction, in Section 2, the spatial scaling law of the kinetic energy spectrum E of velocity $\boldsymbol{u}$ is formulated by using the dimensional analysis, the scaling laws by using Barenblatt's incomplete similarity theory are presented, and some special cases of the laws are obtained. In Section 3, the scaling laws of the density-weighted energy spectrum E_ρ of $\rho^{1/3}\boldsymbol{u}$ are given by the similar way as in Section 2. In Section 4, the relationship between E and E_ρ is discussed. In Section 5, application to a von Weizsäcker's simple compressible cascade model is presented. In Section 6, the discussion on the obtained scaling laws is proposed. Finally, Section 7 concludes the paper with some highlights.

2 Kinetic energy spectrum of compressible turbulence for velocity

In general, a compressible flow deals with the fluid density, which varies significantly in response to a change in the pressure that is caused by a high flow speed. Compressibility effects are typically considered to be significant if the Mach number Ma of the flow exceeds 0.3. For incompressible fluid, there is no need to consider changes in the mass density. However, the mass density change is the central concern for the compressible flow, which must be taken into account in the formulation.

In order to formulate the compressible turbulence scaling law, Sun[22] extended Kolmogorov's assumption from incompressible turbulence to a compressible case. The idea of this extension stemmed from the dimensional analysis of the lift of a wing, in which the lift force F_{L} is a function of the velocity V, the air density ρ, the wing cross-sectional area A, the angle of attack α, the viscosity μ, and the speed of sound c, which will provide the lift $F_{\mathrm{L}} = f(V, A, \rho, \mu, c, \alpha)$. According to the dimensional analysis, the lift is $F_{\mathrm{L}} = \frac{1}{2}\rho V^2 A f(Re, Ma, \alpha)$. This relationship is still valid for any Reynolds number Re and Mach number Ma.

2.1 Dimensional analysis and choice of dimensional variables for compressible turbulence

Any physical relationship can be expressed in a dimensionless form. The implication of this statement is that all of the fundamental equations of physics, as well as all approximations of these equations and, for that matter, all functional relationships between these variables, must be invariant under a dilation of the dimensions of the variables. This is because the variables are subject to measurement by an observer in terms of units that are selected at the arbitrary discretion of the observer. It is clear that a physical event cannot depend on the particular ruler. This principle is the basis for a powerful method of reduction, which is called the dimensional analysis and is useful for the investigation of complicated problems[28–32].

Often, the dimensional analysis is conducted without any explicit consideration for the actual equations that may govern a physical phenomenon. Only the variables that affect the problem are considered. In fact, this is a little deceiving. Inevitably, the choice of the variables is intimately connected to the phenomenon itself. Therefore, it is always connected to the governing equations. The most complex problems in the dimensional analysis tend to be filled with ambiguity, particularly regarding the choice of variables that govern the phenomenon in question. The success or failure of the dimensional analysis depends entirely on the choice of dimensionally physical variables that are relevant to the problem. This constitutes the art of the dimensional analysis. Applying the dimensional analysis intelligently with a deep knowledge of the problem, may yield important and profound results. If applied blindly, dimensional analysis can easily lead to nonsense[28–32].

From a physical point of view, the dimensional analysis is a universal method, which can, of course, be used for the study of compressible turbulence. The difference between incompressible turbulence and its compressible counterpart is that the flow mass density ρ will no longer be a constant, because in the compressible case, the mass density changes as a result of a high speed that will generate shock waves and some of the interactions mentioned in above. Our belief is that no matter how complex the compressible turbulence is, as long as we can capture all the primary variables of the problem, we can formulate it by use of dimensional analysis.

Kolmogorov[9–11] provided some of the most important and most-often quoted results of the incompressible homogeneous turbulence theory. These results comprised what is now referred to as the Kolmogorov's theory in 1941 (K41 theory), and represented a major success of the statistical theories of turbulence. This theory provides a prediction for the energy spectrum of a three-dimensional isotropic homogeneous turbulent flow. Kolmogorov proved that, even though the velocity of an isotropic homogeneous turbulent flow fluctuates in an unpredictable fashion, the energy spectrum (how much kinetic energy is present on average at a particular scale), is predictable.

2.2 Extended Kolmogorov's assumption

In the initial subrange, the K41 theory assumes that the spectrum E, at any particular wavenumber k, depends only on the dissipation rate ε, namely, $E = f(k, \varepsilon)$.

Due to the great success of the Kolmogorov's theory, it would be natural to attempt to extend the Kolmogorov's idea to compressible turbulence. As we know, the basic difference between incompressible and compressible flow concerns the compressibility of mass density ρ. By taking into account the mass density ρ or the Mach number Ma, we can extend Kolmogorov's assumption to compressible turbulence as follows.

Extended Kolmogorov's assumption In the inertial range, the compressible turbulence energy spectrum E is not only controlled by the dissipation rate ε and the wavenumber k, but also by the fluid density ρ (or the Mach number Ma).

In the extended Kolmogorov's assumption, there are two sets of equivalent variables in the formulation, namely, one includes the mass density ρ, and the other has the Mach number Ma.

Set I For the first set, there are six variables, the energy spectrum E, the dissipation rate ε, the wavenumber k, the kinematic viscosity ν, the current mass density ρ, and the local reservoir values ρ_0 of mass density or stagnation density[33].

All primary dimensions of Set I are listed in Table 1.

Table 1 Primary dimension of Set I

E	ν	ε	k	ρ	ρ_0
L^3t^{-2}	L^2t^{-1}	L^2t^{-3}	L^{-1}	mL^{-3}	mL^{-3}

From Ref. [33], in the case of normal shock, the density ratio $\frac{\rho_0}{\rho}$ and the velocity ratio $\frac{v}{c} \equiv Ma$ have a relationship $\frac{\rho_0}{\rho} = ((\gamma+1)Ma^2)/(2+(\gamma-1)Ma^2)$, where $\gamma = \frac{c_p}{c_v}$ is the ratio of heat capacities. This means that the two ratios are dependent, and we can use either $\frac{\rho_0}{\rho}$ or $\frac{v}{c}$ in the formulation. Then, we have Set II of dimensional variables.

Set II For the 2nd set, there are six variables, the energy spectrum E, the dissipation rate ε, the wavenumber k, the kinematic viscosity ν, and the mass density ρ, and its reservoir values can be replaced by the fluid velocity v and the speed of sound c.

All primary dimensions of Set II are listed in Table 2.

Table 2 Primary dimension of Set II

E	ν	ε	k	c	v
L^3t^{-2}	L^2t^{-1}	L^2t^{-3}	L^{-1}	Lt^{-1}	Lt^{-1}

2.3 Scaling laws based on Set I of dimensional variables

According to the Buckingham Π theorem[28], the energy spectrum E can be expressed as the function of $(\nu, \varepsilon, k, \rho_0, \rho)$,

$$E = f(\nu, \varepsilon, k, \rho_0, \rho). \tag{1}$$

Within the six variables, there are three basic units, namely, the time t, the mass m, and the length L. We can choose three repeating variables, namely, the wavenumber k, the dissipation rate ε, and the density ρ, and the dependent variables are the energy spectrum E, the kinematic viscosity ν, and the local reservoir value of density ρ_0.

From the Buckingham Π theorem of dimensional analysis, we can generate three dimensionless variables, namely, $\Pi_1 = E\varepsilon^{-3/2}k^{5/3}$, $\Pi_2 = \nu\varepsilon^{-1/3}k^{4/3}$, and $\Pi_3 = \rho_0\rho^{-1}$. Then, we have the scaling law of the energy spectrum $\Pi_1 = f(\Pi_2, \Pi_3)$, that is,

$$E(k, \varepsilon, \rho) = \varepsilon^{2/3}k^{-5/3}f((k\eta)^{4/3}, \rho_0\rho^{-1}). \tag{2}$$

2.4 Scaling laws based on Set II of dimensional variables

If we use the fluid velocity v and the sound speed c instead of the mass density, we have the second set of variables $(E, \nu, \varepsilon, k, c, v)$, as well as another version of Eq. (1) as follows:

$$E = f(\nu, \varepsilon, k, c, v). \tag{3}$$

Hence, we can have a set of corresponding dimensionless Π which is $\Pi_1 = E\varepsilon^{-3/2}k^{5/3}$, $\Pi_2 = \nu\varepsilon^{-1/3}k^{4/3}$, and $\Pi_3 = v/c = Ma$, where $Ma = v/c$ is the Mach number, and the scaling law of the energy spectrum can be expressed in terms of the Mach number as $\Pi_1 = f(\Pi_2, \Pi_3)$. Therefore,

$$E(k, \varepsilon, Ma) = \varepsilon^{2/3}k^{-5/3}f((k\eta)^{4/3}, Ma). \tag{4}$$

$E(k, \varepsilon, Ma)$ in Eq. (4) is actually equivalent to $E(k, \varepsilon, \rho)$ in Eq. (2), because the ratio of density in Eq. (2) can be expressed in terms of the Mach number. Hence, the energy spectrum equation (2) can be rewritten as the function of Eq. (4).

It should be pointed out that the dimensionless function $f(x, y)$ cannot be completely determined by only the dimensional analysis, which should be finalized by other ways such as numerical simulations or experiments.

2.5 Scaling laws based on incomplete similarity

For incompressible turbulence, we can further simplify Eq. (4). According to the Kolmogorov's assumption, in the inertial subrange, the term $k\eta \to 0$, for the finite value of the Mach number Ma, the function is $f((k\eta)^{4/3}, Ma) \to f(0, Ma)$. Kolmogorov assumed that in the limit $k\eta \to 0$, the function $f(0, x)$ simply assumes the constant value of C_{K}. In other words, there is complete similarity with respect to the variable $k\eta \to 0$, and hence we have $E(k) = C_{\mathrm{K}}\varepsilon^{2/3}k^{-5/3}$. This is the famous Kolmogorov's $-\frac{5}{3}$ power spectrum, which is one of the cornerstones of the turbulence theory. C_{K} is a universal constant, the Kolmogorov's constant, experimentally found to be approximately 1.5.

However, for the compressible turbulence, the existence of the limit of $f(x, Ma)$ as $x \to 0$ is a question owing to intermittency, the fluctuations of the energy-dissipation rate about its mean value ε. According to Barenblatt[30], the incomplete similarity in the variable $k\eta$ would require the nonexistence of a finite and nonzero limit of $f(x, Ma)$ as $x \to 0$.

To obtain more information from Eq. (4), let us use Barenblatt's incomplete similarity theory[30] to simplify the function $f(x, y)$. For this purpose, we propose two hypotheses for compressible turbulence, as shown below.

Hypothesis Ⅰ There is incomplete similarity regarding the energy spectrum $E(k, \varepsilon, Ma)$ in the parameter $k\eta$ and no similarity in the Mach number Ma.

Hypothesis Ⅱ The energy spectrum $E(k, \varepsilon, Ma)$ tends to a well-defined limit as the viscosity tends to be very small but not zero.

In terms of the first hypothesis, for a large Ma, the function $f(x, y)$ could be assumed to be the power function of its argument $k\eta$, while there is no kind of similarity in Ma, as follows:

$$f(k\eta, Ma) = A(Ma)((k\eta)^{4/3})^{B(Ma)}. \tag{5}$$

Then, we arrive at the energy spectrum relationship,

$$E(k, \varepsilon, Ma) = \varepsilon^{2/3}k^{-5/3}A(Ma)((k\eta)^{4/3})^{B(Ma)}, \tag{6}$$

where $A(Ma)$ and $B(Ma)$ are functions of the Mach number Ma.

Using a similar approach proposed by Barenblatt[30], the $B(Ma)$ can be further simplified to a linear form $B(Ma) = \sigma + \beta\epsilon$, where σ and β are adjustable parameters, and the asymptotic parameter ϵ is a function of the Mach number vanishing when $Ma \to \infty$. Then, we have

$$E(k, \varepsilon, Ma) = \varepsilon^{2/3}k^{-5/3}A((k\eta)^{4/3})^{b+\beta\epsilon}. \tag{7}$$

By applying the second hypothesis, a well-defined limit of $E(k, \varepsilon, Ma)$ exists only if $\sigma = 0$. Applying the mathematical identity $x^a = \mathrm{e}^{a\ln x}$ to Eq. (7), we have

$$E(k, \varepsilon, Ma) = \varepsilon^{2/3}k^{-5/3}A\mathrm{e}^{4(\beta\epsilon\ln(k\eta))/3}. \tag{8}$$

From the limit analysis of Eq. (8) when the viscosity vanishes $k\eta \to 0$, Sun[22] proposed $\epsilon = \frac{1}{\ln Ma}$, which can be viewed as the extension of similar result that was firstly obtained by Barenblatt[30] for the boundary turbulence flow, where the small perturbation parameter is $\epsilon = \frac{1}{\ln Re}$, in which Re is the Reynolds number.

However, at $Ma = 1$, the proposed $\epsilon = \frac{1}{\ln Ma}$ has a singularity[22], which should be avoided. In the following, we will try to find another option by taking into account some recent numerical simulations.

It is clear that the parameter ϵ must be a function of the Mach number, let's say, $\epsilon = \xi(Ma)$. Equation (8) would be in the form of

$$E(k, \varepsilon, Ma) = A\varepsilon^{2/3}k^{-5/3}((k\eta)^{4/3})^{\beta\xi(Ma)}. \tag{9}$$

Substituting the Kolmogorov's length $\eta = (\nu^3/\varepsilon)^{1/4}$ into Eq. (9), we have

$$\begin{aligned} E(k,\varepsilon,Ma) &= A\varepsilon^{2/3}k^{-5/3}((k(\nu^3/\varepsilon)^{1/4})^{4/3})^{\beta\xi(Ma)} \\ &= A\varepsilon^{2/3}k^{-5/3}((k(\nu^3/\varepsilon)^{1/3})^{\beta\xi(Ma)} \\ &= A\varepsilon^{2/3}k^{-5/3}(k\nu\varepsilon^{-1/3})^{\beta\xi(Ma)}, \end{aligned} \tag{10}$$

which can be rewritten as a compact form,

$$E(k,\varepsilon,Ma) = C\varepsilon^{(\frac{2}{3}-\frac{1}{3}\beta\xi(Ma))}k^{(-\frac{5}{3}+\beta\xi(Ma))}, \tag{11}$$

where the coefficient $C = A\nu^{\beta\xi(Ma)}$, which appears to be proportional to $\beta\xi(Ma)$.

2.6 Determination of β, $\xi(Ma)$, and C

In Eq. (11), there are two unknowns β and $\xi(Ma)$. The choice of the unknown function $\xi(Ma)$ has some possibilities. From physics of compressible turbulence, we guess that $\xi(Ma)$ should be linked to the shock wave, and it means that the energy spectrum should be affected by the shock wave[1]. In the spirit of the shock wave analysis[33], $\xi(Ma)$ is proposed as $\xi = \ln(\rho_0/\rho)$, namely,

$$\xi(Ma) = \ln\frac{(\gamma+1)Ma^2}{2+(\gamma-1)Ma^2}, \quad 1 \leqslant Ma < \infty. \tag{12}$$

Equation (12) gives

$$\xi(Ma) = \begin{cases} \ln\dfrac{\gamma+1}{\gamma+1} = 0, & Ma = 1, \\ \\ \ln\dfrac{\gamma+1}{\gamma-1}, & Ma \to \infty. \end{cases} \tag{13}$$

Numerical simulations[1,7–8] predicted that the energy spectrum $E \sim k^{h(Ma)}$ tends to be $h = -5/3$ at $Ma = 1$ and tends to be $h = -2$ when $Ma \to \infty$.

It is easy to verify that, at $Ma = 1$, we have $h(Ma) = -\frac{5}{3} + \beta\ln\frac{\gamma+1}{\gamma+1} = -\frac{5}{3}$, which is compatible with the numerical predication. At $Ma \to \infty$, if we set $h(Ma) = -\frac{5}{3}+\beta\ln\frac{\gamma+1}{\gamma-1} = -2$, then we obtain the adjustable parameter β as follows:

$$\beta = -\frac{1}{3}\frac{1}{\ln\frac{\gamma+1}{\gamma-1}}. \tag{14}$$

Within the inertial range, the coefficient C must be independent of the kinematic viscosity ν and Ma, and from numerical simulations[1,7–8], the coefficient C is determined as $C = 1.5$.

With Eqs. (12) and (14), the energy spectrum equation (11) can be expressed as

$$E(k,\varepsilon,Ma) = 1.5\varepsilon^{d(Ma)}k^{h(Ma)}, \tag{15}$$

where the exponents are

$$d(Ma) = \frac{2}{3} + \frac{1}{9}\xi(Ma)\Big(\ln\frac{\gamma+1}{\gamma-1}\Big)^{-1} = \frac{2}{3} + \frac{1}{9}\Big(\ln\frac{\gamma+1}{\gamma-1}\Big)^{-1}\ln\frac{(\gamma+1)Ma^2}{2+(\gamma-1)Ma^2}, \tag{16}$$

$$h(Ma) = -\frac{5}{3} - \frac{1}{3}\xi(Ma)\Big(\ln\frac{\gamma+1}{\gamma-1}\Big)^{-1} = -\frac{5}{3} - \frac{1}{3}\Big(\ln\frac{\gamma+1}{\gamma-1}\Big)^{-1}\ln\frac{(\gamma+1)Ma^2}{2+(\gamma-1)Ma^2}. \tag{17}$$

Equation (15) is the Barenblatt-type incomplete scaling law for compressible turbulence. The formulations, which are presented in this section, provide answers to the first and second questions mentioned in Section 1.

2.7 Scaling laws of some special cases and generalization of Kritsuk's scaling laws

With the modified scaling law in Eq. (11) and/or Eq. (15), we can obtain scaling laws for the following two scenarios: (i) sonic turbulence $Ma = 1$; (ii) hypersonic turbulence $Ma \to \infty$.

(i) For sonic turbulence $Ma = 1$, $\xi(Ma) = \ln \frac{\gamma+1}{\gamma+1} = \ln 1 = 0$, and Eq. (15) is reduced to the $-\frac{5}{3}$ power law as follows:

$$E(k, \varepsilon, Ma)|_{Ma=1} = E(k, \varepsilon, 1) = 1.5\varepsilon^{2/3}k^{-5/3}. \tag{18}$$

(ii) In the case of hypersonic turbulence $Ma \to \infty$, $\xi(Ma) \to \ln \frac{\gamma+1}{\gamma-1}$, then, $\frac{2}{3} + \frac{1}{9}\xi(Ma)$ $(\ln \frac{\gamma+1}{\gamma-1})^{-1} \to \frac{7}{9}$ and $-\frac{5}{3} - \frac{1}{3}\xi(Ma)(\ln \frac{\gamma+1}{\gamma-1})^{-1} \to -2$, Eq. (15) can be reduced to

$$E(k, \varepsilon) = 1.5\varepsilon^{7/9}k^{-2}. \tag{19}$$

k^{-2} is very close to $k^{-1.97}$ numerically obtained by Kritsuk et al.[1] in the case of $Ma = 6$.

To compare with Ref. [1], let $Ma = 6$ in $h(Ma)$. We obtain

$$h(Ma)_{Ma=6} = h(6) = -\frac{5}{3} - \frac{1}{3}\Big(\ln \frac{\gamma+1}{\gamma-1}\Big)^{-1} \ln \frac{\gamma+1}{\gamma-17/18} \approx -\frac{5}{3} - \frac{1}{3} = -2. \tag{20}$$

It means that the proposed $\xi(Ma)$ in Eq. (12) makes the exponent $h(Ma)$ tend to its limit -2 rapidly.

The expression (15) can be viewed as a generalization of Ref. [1]. To appreciate their great contributions, we call the formula (15) Kritsuk's scaling laws.

3 Kinetic energy spectrum of compressible turbulence for density-weighted velocity

Due to the density change of compressible turbulence, as a tradition, the numerical simulations usually use the density-weighted velocity $\boldsymbol{v} = \rho^{1/3}\boldsymbol{u}$ instead of the velocity $\boldsymbol{u}$, where ρ is the density, and $\boldsymbol{u}$ is the velocity.

The corresponding density-weighted energy spectrum E_ρ can be expressed as

$$E_\rho = F(\nu, \varepsilon, k, \rho, Ma). \tag{21}$$

All primary dimensions are listed in Table 3.

Table 3 Primary dimension list

E_ρ	ν	ε	k	ρ	Ma
$m^{1/3}L^2t^{-2}$	L^2t^{-1}	L^2t^{-3}	L^{-1}	mL^{-3}	1

The problem has six parameters and three primary dimensions (m, L, t), and according to the Buckingham Π theorem of dimensional analysis, we will have $\Pi_1 = E_\rho\rho^{-1/3}\varepsilon^{-2/3}k^{5/3}$, $\Pi_2 = (k\eta)^{4/3}$, and $\Pi_3 = Ma$, and their relationship is $\Pi_1 = f(\Pi_2, Ma)$, i.e.,

$$E_\rho(k, \varepsilon, Ma) = \rho^{1/3}\varepsilon^{2/3}k^{-5/3}F((k\eta)^{4/3}, Ma). \tag{22}$$

Equation (22) shows that the density-weighted energy spectrum E_ρ has the same power exponents as E.

In the same way, the Barenblatt's incomplete similarity[30] can also be applied to Eq. (22), which will lead to similar results as before, namely,

$$E_\rho(k, \varepsilon, Ma) = C_\rho\rho^{1/3}\varepsilon^{d(Ma)}k^{h(Ma)}. \tag{23}$$

Wang[19] numerically confirmed that $C_\rho \approx 1.5$, which will be applied to the following formulations. For different values of the Mach number Ma, we can derive some scaling laws as follows:

(i) For sonic turbulence $Ma = 1$ and $\xi(Ma) = \ln 1 = 0$, Eq. (23) is reduced to the $-\frac{5}{3}$ power law as follows:

$$E_\rho(k,\varepsilon,Ma)|_{Ma=1} = E_\rho(k,\varepsilon,1) = 1.5\rho^{1/3}\varepsilon^{2/3}k^{-5/3}. \tag{24}$$

(ii) In the case of highly compressible ($Ma \to \infty$), we have $d(Ma) \to \frac{7}{9}$ and $h(Ma) \to -2$, and Eq. (23) can be simplified to

$$E_\rho(k,\varepsilon,Ma) = 1.5\rho^{1/3}\varepsilon^{7/9}k^{-2}. \tag{25}$$

The scaling law demonstrated in Eq. (24) was confirmed by the numerical simulations by Wang et al.[7] and Wang[19] as illustrated in Fig. 1. The data of Fig. 1 are taken from Ref. [19] and reformatted, where $E_\rho = E(k,\rho)$ is the total kinetic energy, $E_\rho^{\mathrm{c}} = E^{\mathrm{c}}(k,\rho)$ and $E_\rho^{\mathrm{s}} = E^{\mathrm{s}}(k,\rho)$ are compressible and solenoidal parts of $E(k,\rho)$, respectively, and their relationship is $E(k,\rho) = E^{\mathrm{c}}(k,\rho) + E^{\mathrm{s}}(k,\rho)$.

Equation (24) indicates that, with the Mach number close to 1, a density-weighted energy spectrum E_ρ in $k^{-5/3}$ may still be preserved at the small scale ($k\eta \to 0$) if the density-weighted fluid velocity $\boldsymbol{v} = \rho^{1/3}\boldsymbol{u}$ is used. However, for a very large Ma, the energy spectrum E_ρ tends to -2 power laws as stated in Eq. (25). Those have been mentioned by Wang et al.[7].

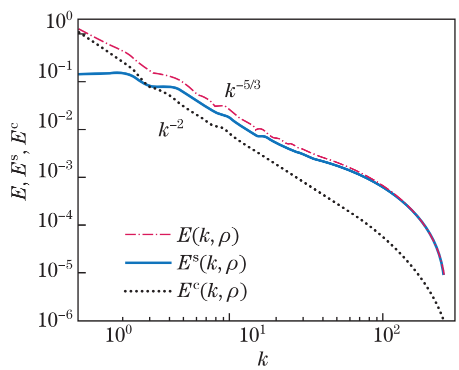

Fig. 1 Density-weighted energy spectrum of compressible turbulence[19]

4 Relationship between E and E_ρ

It might be worth noting that the relationship between the kinetic energy spectrum $E(k,\varepsilon, Ma)$ and its density-weighted counterpart $E_\rho(k,\varepsilon,Ma)$ can be established from Eqs. (2), (4), and (22) as

$$\frac{E_\rho(k,\varepsilon,Ma)}{E(k,\varepsilon,Ma)} = \rho^{1/3}\frac{F((k\eta)^{4/3},Ma)}{f((k\eta)^{4/3},Ma)}, \tag{26}$$

where the functions $F((k\eta)^{4/3},Ma)$ and $f((k\eta)^{4/3},Ma)$ are different from each other, and in general, their ratio is not a constant.

At the small scale ($k\eta \to 0$) and in a limited case of complete compressibility, $Ma \to \infty$, if the ratio in Eq. (26) does exist, then

$$\frac{E_\rho(k,\varepsilon,Ma)}{E(k,\varepsilon,Ma)} \to \lambda\rho^{1/3}. \tag{27}$$

From the numerical simulation[19] with the Mach number close to 1, the coefficient λ can be estimated as $\lambda \simeq 1$.

The relationship (27) answers the question in Section 1. The compressible turbulence can be characterized by the $\rho^{1/3}\boldsymbol{u}$ density-weighted kinetic energy spectrum $E_\rho(k,\varepsilon,Ma)$, as well as by the $\boldsymbol{u}$ kinetic energy spectrum $E(k,\varepsilon,Ma)$.

The formulation in this section shows that the density-weighted kinetic energy spectrum can also be obtained by the dimensional analysis.

5 Application to von Weizsäcker's simple compressible cascade model

In 1951, von Weizsäcker[34] proposed a phenomenological model for three-dimensional compressible turbulence with an intermittent, scale-invariant hierarchy of density fluctuations described by a simple equation that relates the mass density at two successive levels to the corresponding scales through a universal measure of the degree of compression α,

$$\frac{\rho_\nu}{\rho_{\nu-1}} = \Big(\frac{l_\nu}{l_{\nu-1}}\Big)^{-3\alpha}. \tag{28}$$

Here, l_ν is a box size of a sequence ν, and their definition can be found in Ref. [1].

The only free parameter of the model is the geometrical factor α, which takes the value of 1 in a special case of isotropic compression in three dimensions, 1/3 for a perfect one-dimensional compression, and zero in the incompressible limit. The kinetic energy supplied to the system at the large scale is being transferred through the hierarchy by nonlinear interactions. Lighthill[24] pointed out that, in compressible fluid, the mean volume energy transfer rate $\rho u^2 u/l$ is constant in a statistical steady state, so that

$$u \sim (l/\rho)^{1/3}. \tag{29}$$

From these two equations, assuming mass conservation, Fleck[35] derived a set of scaling relationships for the velocity, specific kinetic energy, and density,

$$u \sim l^{1/3+\alpha}, \tag{30}$$

$$E(k) \sim k^h \sim k^{-5/3-2\alpha}, \tag{31}$$

$$\rho \sim l^{-3\alpha}, \tag{32}$$

where all the exponents depend on the compression measure α.

If we compare $E(k) \sim k^h \sim k^{-5/3-2\alpha}$ with our formulation, we can get a relationship $-5/3-2\alpha = h(Ma)$, then, we obtain the compression measure α as follows:

$$\begin{aligned}\alpha(Ma) &= -\frac{1}{6}\xi(Ma)\Big(\ln\frac{\gamma+1}{\gamma-1}\Big)^{-1}\\ &= -\frac{1}{6}\Big(\ln\frac{\gamma+1}{\gamma-1}\Big)^{-1}\ln\frac{(\gamma+1)Ma^2}{2+(\gamma-1)Ma^2}.\end{aligned} \tag{33}$$

From this formula, we can get

$$\alpha = \begin{cases} 0, & Ma = 1,\\ \dfrac{1}{6}, & Ma \to \infty.\end{cases} \tag{34}$$

$\alpha = 1/6 \approx 0.1667$ is close to $\alpha = 0.15$ numerically predicted by Kritsuk et al.[1], which is compatible with his own scaling laws $E(k) \sim k^{-1.97}$. Meneveau and Sreenivasan[36] proposed $\alpha = 1/6$. Equation (33) agrees with all previous predictions.

6 Discussion

This article proposes spatial scaling laws of the kinetic energy spectrum $E(k, \varepsilon, Ma)$ of compressible turbulence flow and its density-weighted counterpart $E_\rho(k, \varepsilon, Ma)$ in terms of the wavenumber k, dissipation rate ε, and Mach number Ma. The study shows that the compressible turbulence kinetic energy spectrum and the density-weighted energy spectrum do not behave in complete similarity, but rather in incomplete similarity, as in Eqs. (11) and (22), which shows that, within the initial subrange, both energy spectra approach the –5/3 and –2 power law of the wavenumber, when the Mach number tends to unity and infinity, respectively.

Why can the energy spectrum in $k^{-5/3}$ still be preserved for $Ma = 1$? As we know, there are nonlinear interactions between solenoidal and compressive modes of velocity fluctuations. The numerical simulations show that the compressive kinetic energy E^{c} and its solenoidal counterpart E^{s} follow different cascade scaling laws[7,19]. The total kinetic energy spectrum can be decomposed into compressible and solenoidal parts, $E = E^{\mathrm{c}} + E^{\mathrm{s}} = c^{\mathrm{c}}k^{-2} + c^{\mathrm{s}}k^{-5/3} = k^{-5/3}(c^{\mathrm{c}}k^{-8/9} + c^{\mathrm{s}})$, in which the compressible energy spectrum $E^{\mathrm{c}} = c^{\mathrm{c}}k^{-2}$ and the solenoidal energy spectrum $E^{\mathrm{s}} = c^{\mathrm{s}}k^{-5/3}$, and c^{c} and c^{s} are constants.

The compressible kinetic energy E^{c} cascade follows the k^{-2} power law of the wavenumber, and the solenoidal kinetic energy E^{s} follows the $k^{-5/3}$ power laws. For the case of $Ma = 1$, the solenoidal E^{s} dominates the energy spectrum, which leads to $E = E^{\mathrm{c}} + E^{\mathrm{s}} = k^{-5/3}(c^{\mathrm{c}}k^{-8/9} + c^{\mathrm{s}}) \approx c^{\mathrm{s}}k^{-5/3}$, and will overall lead the total kinetic energy spectrum E to exhibit a $-\frac{5}{3}$ scaling law as stated by Wang[7]. However, in the case of the large Mach number, the compressive kinetic energy E^{c} will dominate the process of cascade. Then, it will lead to $E = E^{\mathrm{c}} + E^{\mathrm{s}} = k^{-2}(c^{\mathrm{c}} + c^{\mathrm{s}}k^{1/3}) \approx c^{\mathrm{s}}k^{-2}$.

As stated in the extended Kolmogorov's assumption, all formulations in the paper are only valid within the inertial range. In other words, the limitation to all of the proposed spatial energy spectra is that the formulations of the energy spectra deduced by the dimensional analysis and Barenblatt's incomplete similarity theory can only apply to the inertial scale ranges rather than the whole range of small scales (especially when $k\eta$ tends to zero). The pivotal results obtained in this article, i.e., the several "general" expressions of the energy spectrum, namely, Eqs. (15), (23) and others can at most apply to the inertial range rather than the dissipation range because the spectrum decays more rapidly or even at an exponential rate in the smaller dissipation scales, i.e., evidently this fact cannot be covered in those formulae of the spatial energy spectrum.

7 Conclusions

Theoretically, this paper proposes the scaling laws for compressible turbulence flow by using Barenblatt's incomplete similarity theory. The spatial scaling laws of velocity kinetic energy spectrum for compressible turbulence flow and its density-weighted counterpart have been formulated in terms of the wavenumber, dissipation rate, and Mach number by using the dimensional analysis. The results show that, within the initial subrange, both energy spectra approach the –5/3 and –2 power law of the wavenumber, when the Mach number tends to unity and infinity, respectively.

The predicted relationships may benefit the understanding of compressible turbulence. It might be worth mentioning that the scaling laws proposed in this article should be verified again by experiments and numerical simulations. The methodology proposed in this paper can be used for other problems, such as the temporal scaling laws of turbulence[37].

For easy using purpose, the main results are highlighted in Table 4.

Table 4 Kinetic spectrum scaling law of compressible turbulence

Item	Description
$\boldsymbol{u}$ spectrum E	$E = E(k, \varepsilon, Ma) = 1.5\varepsilon^{d(Ma)}k^{h(Ma)}$
$\rho^{1/3}\boldsymbol{u}$ spectrum E_ρ	$E_\rho = 1.5\rho^{1/3}E(k, \varepsilon, Ma) = 1.5\rho^{1/3}\varepsilon^{d(Ma)}k^{h(Ma)}$
$d(Ma),\ Ma \in [0, 1]$	$\dfrac{2}{3}$
$h(Ma),\ Ma \in [0, 1]$	$-\dfrac{5}{3}$
$d(Ma),\ Ma \in [1, \infty)$	$\dfrac{2}{3} + \dfrac{1}{9}\Big(\ln\dfrac{\gamma+1}{\gamma-1}\Big)^{-1}\ln\dfrac{(\gamma+1)Ma^2}{2+(\gamma-1)Ma^2}$
$h(Ma),\ Ma \in [1, \infty)$	$-\dfrac{5}{3} - \dfrac{1}{3}\Big(\ln\dfrac{\gamma+1}{\gamma-1}\Big)^{-1}\ln\dfrac{(\gamma+1)Ma^2}{2+(\gamma-1)Ma^2}$
Limitation	Inertial range

Acknowledgements This paper commemorates my beloved father, Zhongchuan SUN. Supports from the South Africa National Research Foundation (NRF), Cape Peninsula University of Technology (CPUT), and the State Key Laboratory for Turbulence and Complex Systems at Peking University are gratefully acknowledged. The author would like to express his most sincere thanks to reviewers for their high level academic comments and corrections, and their professionalism inspires me deeply. The author wishes to take this opportunity to appreciate the general discussion on turbulence with Professors Shiyi CHEN, Zhensu SHE, Cunbiao LEE, Yipeng SHI, and Jiancun WANG.

References

[1] Kritsuk, A. G., Norman, M. L., Padoan, P., and Wagner, R. The statistics of supersonic isothermal turbulence. *The Astrophysical Journal*, **665**, 416–431 (2007)

[2] Frisch, U. *Turbulence*: *The Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge (2008)

[3] Lee, C. B. and Wu, J. Z. Transition in wall-bounded flows. *Applied Mechanics Reviews*, **61**, 030802 (2008)

[4] She, Z. S., Chen, X., Wu, Y., and Hussain, F. New perspective in statistical modeling of wall-bounded turbulence. *Acta Mechanica Sinica*, **26**, 847–861 (2010)

[5] Zhang, Y. S., Bi, W. T., Hussain, F., Li, X. L., and She, Z. S. Mach-number-invariant mean-velocity profile of compressible turbulent boundary layers. *Physical Review Letters*, **109**, 054502 (2012)

[6] Zhou, H. and Zhang, H. X. What is the essence of the so-called century lasting difficult problem in classic physics, the “problem of turbulence”? *Scienta Sinica*: *Physica*, *Mechanica and Astronomica*, **42**, 1–5 (2012)

[7] Wang, J., Yang, Y., Shi, Y., Xiao, Z., He, X. T., and Chen, S. Cascade of kinetic energy in three-dimensional compressible turbulence. *Physical Review Letters*, **110**, 214505 (2013)

[8] Chen, S. Y., Xia, Z. H., Wang, J. C., and Yang, Y. T. Recent progress in compressible turbulence. *Acta Mechanica Sinica*, **31**, 275–291 (2015)

[9] Kolmogorov, A. N. The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. *Proceedings of the Royal Society of London*, **434**, 9–13 (1991)

[10] Kolmogorov, A. N. On degeneration (decay) of isotropic turbulence in an incompressible visous liquid. *Doklady Akademii Nauk SSSR*, **31**, 538–540 (1941)

[11] Kolmogorov, A. N. Dissipation of energy in locally isotropic turbulence. *Proceedings of the Royal Society of London*, **434**, 15–17 (1991)

[12] Aluie, H. Compressible turbulence: the cascade and its locality. *Physical Review Letters*, **106**, 174502 (2011)

[13] Aluie, H., Li, S., and Li, H. Conservative cascade of kinetic energy in compressible turbulence. *Astrophysical Journal Letters*, **751**, L29 (2012)

[14] Aluie, H. Scale decomposition in compressible turbulence. *Physica D: Nonlinear Phenomena*, **247**, 54–65 (2013)

[15] Armstrong, J. W., Spangler, S. R., and Rickett, B. J. Electron density power spectrum in the local interstellar medium. *The Astrophysical Journal*, **443**, 209–221 (1995)

[16] Cardy, J., Falkovich, G., and Gawedzki, K. *Nonequilibrium Statistical Mechanics and Turbulence*, Cambridge University Press, Cambridge (2008)

[17] Chu, B. T. and Kovasznay, L. S. G. Non-linear interactions in a viscous heatconducting compressible gas. *Journal of Fluid Mechanics*, **3**, 494–514 (1958)

[18] Federrath, C., Roman-Duval, J., Klessen, R. S., Schmidt, W., and Mac Low, M. M. Comparing the statistics of interstellar turbulence in simulations and observations: solenoidal versus compressive turbulence forcing. *Astronomy and Astrophysics*, **512**, A81 (2010)

[19] Wang, J. *Cascade of Kinetic Energy and Thermodynamic Process in Compressible Turbulence* (in Chinese), Post-Doctoral Research Report, Peking University (2014)

[20] Schmidt, W., Federrath, C., and Klessen, R. Is the scaling of supersonic turbulence universal? *Physical Review Letters*, **101**, 194505 (2008)

[21] Galtier, S. and Banerjee, S. Exact relation for correlation functions in compressible isothermal turbulence. *Physical Review Letters*, **107**, 134501 (2011)

[22] Sun, B. H. The spatial scaling laws of compressible turbulence. https://arxiv.org/abs/1502.02815v5 (2016)

[23] Kovasznay, L. S. G. Turbulence in supersonic flow. *Journal of the Aeronautical Sciences*, **20**, 657–674 (1953)

[24] Lighthill, M. J. The effect of compressibility on turbulence. *Proceedings from the 2nd International Astronomical Union Symposium on Gas Dynamics of Cosmic Clouds*, North Holland Publishing Company, Amsterdam (1955)

[25] Moiseev, S. S., Toor, A. V., and Yanovsky, V. V. The decay of turbulence in the Burgers model. *Physica D: Nonlinear Phenomena*, **2**, 187–193 (1981)

[26] Kadomtsev, B. B. and Petviashvili, V. I. Acoustic turbulence. *Soviet Physics Doklady*, **18**, 115–118 (1973)

[27] Shivamoggi, B. K. Multifractal aspects of the scaling laws in fully developed compressible turbulence. *Annals of Physics*, **243**, 169–176 (1995)

[28] Bridgman, P. W. *Dimensional Analysis*, Yale University Press, New Haven (1922)

[29] Sedov, L. I. *Similarity and Dimensional Analysis in Mechanics*, Academic Press, New York (1959)

[30] Barenblatt, G. I. *Similarity, Self-similarity and Intermediate Asymptotics*, Cambridge University Press, Cambridge (1996)

[31] Cantwell, B. J. *Introduction to Symmetry Analysis*, Cambridge University Press, Cambridge (2002)

[32] Sun, B. H. *Dimensional Analysis and Lie Group* (in Chinese), China High Education Press, Bejing (2016)

[33] Liepmann, H. W. and Roshko, A. *Elements of Gasdynamics*, Dover Publications, New York (1993)

[34] Von Weizsäcker, C. F. The evolution of galaxies and stars. *The Astrophysical Journal*, **114**, 165–186 (1951)

[35] Fleck, R. C., Jr. Scaling relations for the turbulenct, non-self-gravitating, neutral component of the interstellar medium. *The Astrophysical Journal*, **458**, 739–741 (1996)

[36] Meneveau, C. and Sreenivasan, K. R. Interface dimension in intermittent turbulence. *Physical Review A*, **41**, 2246–2248 (1990)

[37] Sun, B. H. The temporal scaling laws of compressible turbulence. *Modern Physics Letters B*, **30**, 1650297 (2016)