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Corrigendum: incompatible deformation field and Riemann curvature tensor*

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Abstract The "Corollary 1" formulation in SUN, B. H. Incompatible deformation field and Riemann curvature tensor. *Applied Mathematics and Mechanics* (*English Edition*), **38**(3), 311–332 (2017) is corrected. It can be stated as follows: The symmetric part of the deformation gradient has no contribution to the trace of the displacement density tensor.

Key words Riemann curvature tensor, deformation gradient, displacement density tensor

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Nomenclature

$\boldsymbol{X}, \boldsymbol{Y}$,	position vectors in the undeformed state;	$\boldsymbol{\nabla}$,	gradient of ∇;
$\boldsymbol{x}, \boldsymbol{y}$,	position vectors in the deformed state;	$\boldsymbol{\nabla}_{\boldsymbol{X}}$,	gradient of ∇ respect to $\boldsymbol{X}$;
$\boldsymbol{G}_A$,	base vector in the undeformed state;	$\boldsymbol{\nabla}_{\boldsymbol{x}}$,	gradient of ∇ respect to $\boldsymbol{x}$;
$\boldsymbol{g}_k$,	base vector in the deformed state;	$\boldsymbol{\varepsilon}$,	permutation tensor;
$\boldsymbol{u}$,	displacement vector;	$\boldsymbol{\Omega}$,	antisymmetric part of the displacement gradient;
$\boldsymbol{F}$,	deformation gradient;	$\otimes$,	tensor product;
$\boldsymbol{R}$,	Riemann curvature tensor in the undeformed state;	$\boldsymbol{I}$,	Kronecker delta, $\boldsymbol{I} = \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$;
$\Delta\boldsymbol{u}$,	displacement vector variation;	$\boldsymbol{T}$,	displacement flux density tensor.

The "Corollary 1" formulation in Ref. [1] is incorrectly presented. The flawless should be maintained, and the misrepresentation must be corrected.

1 Corollary 1 and its proof in Ref. [1]

In Ref. [1], Corollary 1 is incorrectly proposed. Making the letter self-contained, the corollary and its proof are rewritten as follows:

Corollary 1 *The symmetric part of the deformation gradient* $\boldsymbol{F}$ *has no contribution to the displacement change* $\Delta\boldsymbol{u}$ *and the displacement density tensor* $\boldsymbol{T}$.

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Proof Let $\boldsymbol{S} = \frac{1}{2}(\boldsymbol{F}+\boldsymbol{F}^{\mathrm{T}})$ and $\boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{F}-\boldsymbol{F}^{\mathrm{T}})$ be the symmetric part and anti-symmetric part of the deformation gradient $\boldsymbol{F}$, respectively. As we know, any tensor can be decomposed into a symmetric part and an antisymmetric part, i.e., $\boldsymbol{F} = \boldsymbol{S} + \boldsymbol{\Omega}$. Since the curl of the symmetric tensor vanishes, $\mathrm{curl}\boldsymbol{S} = \boldsymbol{S} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{0}$, and $\mathrm{curl}\boldsymbol{F} = \mathrm{curl}(\boldsymbol{S}+\boldsymbol{\Omega}) = \mathrm{curl}\boldsymbol{\Omega} = \boldsymbol{\Omega} \times \boldsymbol{\nabla}_{\boldsymbol{X}}$. Then, we have

$$\mathrm{curl}\boldsymbol{\Omega} = \mathrm{curl}\Big(\frac{1}{2}(\boldsymbol{F}-\boldsymbol{F}^{\mathrm{T}})\Big) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} = \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}. \tag{1}$$

$\mathrm{curl}\boldsymbol{S} = \boldsymbol{0}$ indicates that the symmetric part of the deformation gradient $\boldsymbol{F}$ does not have any contribution towards to the comparability conditions. In other words, the symmetric deformations are always compatible, and the incompatible deformation will make the symmetric deformation breaks down.

Since $\mathrm{curl}\boldsymbol{\Omega} = \boldsymbol{\Omega} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{\Omega}\boldsymbol{\nabla}_{\boldsymbol{X}} : \boldsymbol{\varepsilon}$ and thus $\boldsymbol{\Omega}\boldsymbol{\nabla}_{\boldsymbol{X}} : \boldsymbol{\varepsilon} = \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} : \boldsymbol{\varepsilon}$, we have

$$\boldsymbol{\Omega}\boldsymbol{\nabla}_{\boldsymbol{X}} = \frac{1}{2}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} = \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R}. \tag{2}$$

It must be pointed out that the statement of the above corollary and its proof are wrong and should be corrected.

2 Correction of Corollary 1 and its proof

Before presenting the corrected version of the above corollary, let us first prove a Lemma as follows:

Lemma 1 *The trace of curl of a symmetric tensor is zero.*

Proof Let $\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ be a symmetric tensor, i.e., $A_{ij} = A_{ji}$. The curl of the tensor $\boldsymbol{A}$ is defined as follows:

$$\mathrm{curl}\boldsymbol{A} = \boldsymbol{A} \times \boldsymbol{\nabla} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \times \nabla_k \boldsymbol{e}_k = \nabla_k(A_{ij})\boldsymbol{e}_i \otimes \boldsymbol{e}_j \times \boldsymbol{e}_k = \nabla_k A_{ij} e_{jkm} \boldsymbol{e}_i \otimes \boldsymbol{e}_m,$$

where e_{jkm} is the permutation symbol. Therefore, we can define the trace of the curl of tensor $\boldsymbol{A}$ by $\mathrm{tr}(\mathrm{curl}\boldsymbol{A}) = \boldsymbol{I} : \mathrm{curl}\boldsymbol{A} = \boldsymbol{I} : (\boldsymbol{A} \times \boldsymbol{\nabla})$, where $\boldsymbol{I} = \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{e}_i \otimes \boldsymbol{e}_i$ is the Kronecker delta. Therefore,

$$\begin{aligned}
\mathrm{tr}(\mathrm{curl}\boldsymbol{A}) &= \boldsymbol{I} : (\nabla_k A_{ij} e_{jkm} \boldsymbol{e}_i \otimes \boldsymbol{e}_m) \\
&= \delta_{\mu\nu} \boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu : \nabla_k A_{ij} e_{jkm} \boldsymbol{e}_i \otimes \boldsymbol{e}_m \\
&= \delta_{\mu\nu} \nabla_k A_{ij} e_{jkm} (\boldsymbol{e}_\mu \cdot \boldsymbol{e}_i)(\boldsymbol{e}_\nu \cdot \boldsymbol{e}_m) \\
&= \delta_{\mu\nu} \nabla_k A_{ij} e_{jkm} \delta_{\mu i} \delta_{\nu m} \\
&= \delta_{im} \nabla_k A_{ij} e_{jkm} \\
&= \nabla_k A_{ij} e_{jki}.
\end{aligned}$$

Since $A_{ij} = A_{ji}$ and $e_{jki} = -e_{ikj}$, we have $\mathrm{tr}(\mathrm{curl}\boldsymbol{A}) = \nabla_k A_{ij} e_{jki} = 0$.

Corollary 2 *The symmetric part of the deformation gradient* $\boldsymbol{F}$ *has no contribution to the trace of displacement density tensor* $\boldsymbol{T}$.

Proof Let $\boldsymbol{S} = \frac{1}{2}(\boldsymbol{F}+\boldsymbol{F}^{\mathrm{T}})$ and $\boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{F}-\boldsymbol{F}^{\mathrm{T}})$ be the symmetric part and the anti-symmetric part of the deformation gradient $\boldsymbol{F}$, respectively. As we know, any tensor can be decomposed into a symmetric part and an antisymmetric part, i.e., $\boldsymbol{F} = \boldsymbol{S} + \boldsymbol{\Omega}$.

Since the trace of the curl of the symmetric tensor vanishes, i.e., $\mathrm{tr}(\mathrm{curl}\,\boldsymbol{S}) = \mathrm{tr}(\boldsymbol{S} \times \boldsymbol{\nabla}_{\boldsymbol{X}}) = 0$, we have $\mathrm{tr}(\mathrm{curl}\,\boldsymbol{F}) = \mathrm{tr}(\mathrm{curl}(\boldsymbol{S}+\boldsymbol{\Omega})) = \mathrm{tr}(\mathrm{curl}\boldsymbol{S}) + \mathrm{curl}\boldsymbol{\Omega} = \mathrm{tr}(\boldsymbol{\Omega} \times \boldsymbol{\nabla}_{\boldsymbol{X}})$. Then, we have the trace of the displacement density tensor $\boldsymbol{T}$ as follows:

$$\mathrm{tr}\boldsymbol{T} = -\mathrm{tr}(\boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}}) = -\mathrm{tr}(\boldsymbol{\Omega} \times \boldsymbol{\nabla}_{\boldsymbol{X}}) = -\frac{1}{2}\mathrm{tr}((\boldsymbol{F}-\boldsymbol{F}^{\mathrm{T}}) \times \boldsymbol{\nabla}_{\boldsymbol{X}}) = -\frac{1}{2}\mathrm{tr}(\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}). \tag{3}$$

References

[1] SUN, B. H. Incompatible deformation field and Riemann curvature tensor. *Applied Mathematics and Mechanics* (*English Edition*), **38**(3), 311–332 (2017) https://doi.org/10.1007/s10483-017-2176-8