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Incompatible deformation field and Riemann curvature tensor*

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Abstract Compatibility conditions of a deformation field in continuum mechanics have been revisited via two different routes. One is to use the deformation gradient, and the other is a pure geometric one. Variations of the displacement vector and the displacement density tensor are obtained explicitly in terms of the Riemannian curvature tensor. The explicit relations reconfirm that the compatibility condition is equivalent to the vanishing of the Riemann curvature tensor and reveals the non-Euclidean nature of the space in which the dislocated continuum is imbedded. Comparisons with the theory of Kröner and Le-Stumpf are provided.

Key words compatibility condition, Riemann curvature tensor, deformation gradient, Burgers vector, dislocation density tensor

Chinese Library Classification O33, O183, O186
2010 Mathematics Subject Classification 74A05

Nomenclature

$\mathfrak{B}$,	undeformed state or configuration;
$\mathfrak{b}$,	deformed state or configuration;
$\boldsymbol{X}, \boldsymbol{Y}$,	position vector in undeformed state;
$\boldsymbol{x}, \boldsymbol{y}$,	position vector in deformed state;
$\mathrm{d}S$,	line element length in undeformed state;
$\mathrm{d}s$,	line element length in deformed state;
$\boldsymbol{G}_A$,	base vector in undeformed state;
$\boldsymbol{g}_k$,	base vector in deformed state;
G_{AB},	metric tensor in undeformed state;
g_{ij},	metric tensor in deformed state;
$\boldsymbol{u}$,	displacement vector;
$\boldsymbol{F}$,	deformation gradient;
$\boldsymbol{B}$,	left Cauchy-Green deformation tensor;
$\boldsymbol{C}$,	right Cauchy-Green deformation tensor;
$\boldsymbol{E}$,	Green strain tensor;
$\boldsymbol{L}$,	velocity gradient tensor in undeformed state;
$\boldsymbol{l}$,	velocity gradient tensor in deformed state;
$\boldsymbol{D}$,	rate of deformation tensor in undeformed state;
$\boldsymbol{d}$,	rate of deformation tensor in deformed state;
$\boldsymbol{W}$,	spin tensor in undeformed state;
$\boldsymbol{w}$,	spin tensor in deformed state;
R_{KLMN},	Riemann curvature tensor components in undeformed state;
r_{klmn},	Riemann curvature tensor components in deformed state;
$\boldsymbol{R}$,	Riemann curvature tensor in undeformed state;
$\boldsymbol{r}$,	Riemann curvature tensor in deformed state;
Γ^I_{JKL},	Christoffel symbols in undeformed state;
γ^i_{jkl},	Christoffel symbols in deformed state;

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$\Delta \boldsymbol{u}$,	displacement vector variation;
$\boldsymbol{\nabla}$,	gradient nabla;
$\boldsymbol{\nabla}_{\boldsymbol{x}}$,	gradient nabla respect to $\boldsymbol{x}$;
$\partial\Psi$,	boundary of surface Ψ in $\mathfrak{B}$;
$\partial\psi$,	boundary of surface Ψ in $\mathfrak{b}$;
$\boldsymbol{\varepsilon}$,	permutation tensor;
$\boldsymbol{\epsilon}$,	symmetric part of displacement gradient;
$\boldsymbol{\Omega}$,	antisymmetric part of displacement gradient;
$\boldsymbol{\omega}$,	axial vector of $\boldsymbol{\Omega}$;
$\wedge$,	exterior product or wedge product;
$\otimes$,	tensor product;
$\mathrm{d}\boldsymbol{A}$,	area element in undeformed state;
$\mathrm{d}\boldsymbol{a}$,	area element in deformed state;
$\boldsymbol{b}$,	Burgers vector;
$\boldsymbol{T}$,	dislocation density tensor in undeformed state.

1 Introduction

In the continuum description of a solid body, we imagine the body to be composed of a set of infinitesimal volumes or material points. Each volume is assumed to be connected to its neighbours without any gaps or overlaps. Certain mathematical conditions have to be satisfied to ensure that gaps/overlaps do not develop when a continuum body is deformed. These compatibility conditions[1–4] are mathematical conditions that determine whether a particular deformation will leave a body in a compatible state. The investigation of the compatibility conditions will be beneficial to the studies of plastic deformation[5–7] and dislocation/defects[8–22] in the solid .

Love[23] credited Barré de Saint-Venant (1864) who was the first to discover the derivation of the "bulk" compatibility equations. In 1876, the proof of the equations was developed by Kirchhoff[24], and was later rigorously proven by Beltrami[25] in 1886. In 1899, Michell[26] studied the compatibility equations of linearized elasticity in two dimensions for non-simply-connected bodies. He showed that compatibility requires vanishing of certain integrals on each "independent irreducible circuit". In 1901, Weingarten[27] provided a famous theorem on the conservation of the integration of displacement and rotation along any closed loop in the infinitesimal deformation. Cesàro[28] and Volterra[29] studied compatibility equations for non-simply-connected bodies and the possibility of multi-valuedness of displacements when the body is not simply-connected. Volterra[30] may have been the first person to provide the correct definition of dislocation and disclination. Love[23], Krutkov[31], Beltrami[32], Green and Zerna[33], and Seugling[34] realized that the classical compatibility equations of elasticity can be written as vanishing of the curvature tensor of the Levi-Civita connection of strain (understood as a metric). In 1960, Kröner's deep insight was understanding the incompatibility as a genuine geometric property of the dislocated crystal[12]. Pietraszkiewicz[35] and Pietraszkiewicz and Badur[36] studied the problem of calculating the deformation mapping when the right Cauchy-Green strain was given, and they obtained a nonlinear analogue of the Cesàro integral. Blume[37] discussed the compatibility equations in terms of the left Cauchy-Green strain $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ in two dimensions. In 1994 and 1995, Le and Stumpf[18–19] studied the compatibility conditions of elasto-plastic deformation and for the first time obtained the dislocation density tensor in terms of the Christoffel symbols for both elastic and plastic deformation gradients. In 1999, Acharya[38] studied the same problem in three dimensions. Yavari and Goriely[39] and Yavari[40] presented the compatibility conditions in the most abstract format by using exterior differential forms for both simply- and non-simply-connected bodies. It is noteworthy that for the sake of disseminating knowledge of compatibility theory, Guo and Liang[16] wrote the first and only comprehensive monograph on the topics, where they applied abstract tensors, modern non-Riemanian geometry and gauge theory to attack dislocations and defects. In 2016, Sun[17] revisited the explicit expression of incompatible condition in terms of the Riemann curvature tensor and its application to the theory of shells.

In a three-dimensional space, the deformation tensor $\boldsymbol{C} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ and the strain tensor $\boldsymbol{E} = E_{AB}\boldsymbol{G}^A\boldsymbol{G}^B$ both have six components, which are expressible in terms of the three components u_k of the displacement vector, namely, $2\boldsymbol{E} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} - \boldsymbol{I} = \boldsymbol{C} - \boldsymbol{I} = (u_{A;B} + u_{B;A} + u^K_{;A}u^K_{;B})\boldsymbol{G}^A\boldsymbol{G}^B$.

If the displacement vector and components $\boldsymbol{u} = u_A \boldsymbol{G}^A$ possess continuous first-order partial derivatives, we obtain the strain components $2E_{AB} = u_{A;B} + u_{B;A} + u^K_{;A} u^K_{;B}$. Conversely, if the six strains E_{ij} are given, then a question arises regarding the existence of a single-valued continuous displacement field, which corresponds to the given strains. It is clear that the six partial differential equations are an over-determined system, and may not possess such a solution to the three unknown u_A, unless certain integrability conditions are satisfied. These conditions constitute a set of partial differential equations, which involve strains E_{AB} alone, and are known as the compatibility conditions. When the compatibility conditions are violated, the corresponding displacement field is not unique. The body may then possess dislocations.

From a mathematical point of view, a compatible deformation (or strain) tensor field in a body is the one that a unique field is obtained when the body is subjected to a continuous, single-valued, displacement field. Compatibility is the study of the conditions under which such a displacement field can be guaranteed. There are two approaches to finding the compatibility conditions.

An obvious way of finding the compatibility conditions is elimination of the displacements u_A from the six equations $2E_{AB} = u_{A;B} + u_{B;A} + u^K_{;A} u^K_{;B}$ by partial differentiation. This method (for the finite strain) is, however, tedious, if not extremely awkward. There are two alternative methods, namely, the Riemann method, where one makes use of the Riemann theorem, and the other is that the displacement change along a closed loop must be vanishing, which is called the displacement change method.

With the Riemannian method, we know that the three-dimensional space in which the deformation takes place is Euclidean. These arc lengths $\mathrm{d}S$ and $\mathrm{d}s$ of the undeformed and deformed bodies are given by $\mathrm{d}S^2 = \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X} = \delta_{KL} \mathrm{d}X_K \mathrm{d}X_L = g_{kl} \mathrm{d}x_k \mathrm{d}x_l$ and $\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} = \delta_{kl} \mathrm{d}x_k \mathrm{d}x_l = G_{KL} \mathrm{d}X_K \mathrm{d}X_L$. Both the undeformed and deformed bodies are imbedded in an Euclidean space. In the coordinates x_K, the original length is calculated when all six components g_{kl} are known, and for the calculation of the final length, we need all six components of G_{KL}. Thus, if we look at the motions $x_k = x_k(X^1, X^2, X^3)$, $k = 1, 2, 3$, as transformed coordinates from rectangular coordinates X^K to curvilinear coordinates x^k at the fixed time t, then g_{kl} plays the role of metric tensor in the curvilinear coordinates x_k, and the same is valid for G_{KL} for the inverse motion.

In an Euclidean space, any six quantities cannot be a metric tensor unless they satisfy the Riemann theorem[2,16], namely, for a symmetric tensor a_{kl} to be a metric tensor for an Euclidean space, it is necessary and sufficient that a_{kl} should be a nonsingular positive definite tensor, and the Riemann-Christoffel tensor[33,42] R_{KLMN} that is formed from it should vanish identically. Both G_{KL} and g_{kl} are nonsingular symmetric and positive definite tensors of the Euclidean three-dimensional space. Therefore, both the Riemann-Christoffel tensors of the undeformed and deformed configuration must vanish, i.e.,

$$\begin{aligned} R_{KLMN} &= 0, \quad \text{undeformed state}, \\ r_{klmn} &= 0, \quad \text{deformed state}. \end{aligned}$$

Regarding the former condition, the partial differentiation is understood to be with respect to X^K and regarding the latter, with respect to x_k. The $R_{KLMN} = 0$ gives six partial differential equations, which constitute the compatibility conditions for G_{KL}, and $r_{klmn} = 0$ gives six partial differential equations for g_{kl}. Since we have $G_{KL} = \delta_{KL} + 2E_{KL}$ and $g_{kl} = \delta_{kl} - 2e_{kl}$, where E_{KL} is the Lagrange strain tensor, and e_{kl} is the Euler strain tensor, we get compatibility conditions for E_{KL} and e_{kl}. The detailed compatibility conditions for the finite deformation can be found in the master works of Truesdell and Toupin[2].

In the context of the infinitesimal strain theory, these conditions are equivalent to stating that the displacements in a body can be obtained by integrating the strains. Such an integration is possible if the Saint-Venant's tensor vanishes in a simply-connected body, $\boldsymbol{\nabla} \times \boldsymbol{\epsilon} \times \boldsymbol{\nabla} = \boldsymbol{0}$, where $\boldsymbol{\epsilon}$ is the infinitesimal strain tensor. This total expression[1] of compatibility conditions was obtained by Krutkov[31] and Beltrami[32], and was re-introduced by Kröner in the linear continuum theory of dislocation[43].

With the displacement change method, for finite deformations, the compatibility conditions of a simply-connected body can be expressed as the displacement change $\Delta \boldsymbol{u}$ along any closed loop within the body that must vanish, which means that $\Delta \boldsymbol{u} = \oint_{\partial\Psi} \mathrm{d}\boldsymbol{u} = 0$, and according to the Stokes theorem, we have $\oint_{\partial\Psi} \mathrm{d}\boldsymbol{u} = -\int_{\Psi} \boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{A} = 0$, $\partial\Psi$ is the closed boundary of domain or surface Ψ. The integral can be transferred into the form $\mathrm{curl}\boldsymbol{F} = \boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{0}$ or in the differential form $\mathrm{d}\boldsymbol{F} = \boldsymbol{0}$, where $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{x}\boldsymbol{\nabla}_{\boldsymbol{X}}$ is the deformation gradient, and $\boldsymbol{X}$ and $\boldsymbol{x}$ are coordinates in the reference and current configuration, respectively. For non-simply-connected bodies, the compatibility conditions should be $\mathrm{d}\boldsymbol{F} = \boldsymbol{0}$ and $\oint_{\partial\Psi_K} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} = \boldsymbol{0}$, where $\partial\Psi_K$ is the closed loop around the holes, and K runs from 1 to the number of holes or defects[39–40].

Regarding the Riemannian method, it is clear to see that the relationship between the displacement vector variation $\Delta \boldsymbol{u}$ and the compatibility condition $\boldsymbol{R} = \boldsymbol{0}$ is still unknown. Similar to the displacement change method, the explicit relation between the compatibility condition $\boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{0}$ and $\boldsymbol{R}$ has not been established yet. The two methods could be proven to be equivalent if the displacement vector variation $\Delta \boldsymbol{u}$ can be represented in terms of the Riemannian curvature tensor $\boldsymbol{R}$.

The open problem is how to find the relationship between $\Delta \boldsymbol{u}$ and $\boldsymbol{R}$ via either Route 1 or Route 2, where Route 1 uses the deformation gradient, $\Delta \boldsymbol{u} = \oint (\boldsymbol{F} - \boldsymbol{I}) \cdot \mathrm{d}\boldsymbol{X}$, and Route 2 uses a pure geometric formulation, $\Delta \boldsymbol{u} = \oint \mathrm{d}\boldsymbol{u}$. In other words, Route 1 is the deformation kinematics way, and Route 2 is the pure differential geometry way.

For the purpose of clarity, the problem of finding $\Delta \boldsymbol{u} = \underbrace{\oint (\boldsymbol{F} - \boldsymbol{I}) \cdot \mathrm{d}\boldsymbol{X}}_{\text{Route 1}} = \underbrace{\oint \mathrm{d}\boldsymbol{u}}_{\text{Route 2}}$ is shown in Fig. 1.

$$
\begin{tikzcd}
\Delta \boldsymbol{u} \arrow[r, "\text{Route 2}"] \arrow[d, "\text{Route 1}"'] & \oint \mathrm{d}\boldsymbol{u} \arrow[d] \arrow[dl, dotted, leftrightarrow] \\
\oint (\boldsymbol{F} - \boldsymbol{I}) \cdot \mathrm{d}\boldsymbol{X} \arrow[r] & \text{Relationship with } \boldsymbol{R}
\end{tikzcd}
$$

Fig. 1 Two routes to finding $\Delta \boldsymbol{u}$

The aim of this paper is to formulate the explicit expressions of displacement vector variation and other quantities in terms of the Riemann tensor. The paper is organized as follows. Following an introduction, Section 2 derives the displacement vector variation $\Delta \boldsymbol{u}$ in terms of the Riemann tensor $\boldsymbol{R}$, and gives the curl of the deformation gradient, the displacement flux tensor and the rate of deformation tensor. Section 3 reformulates the displacement vector variation by using exterior differential forms. Section 4 presents the Burgers vector and dislocation density tensor and compares with some well-known results. Section 5 provides a proof for the displacement vector variation without using the deformation gradient. Section 6 concludes the paper, and the appendix gives some preliminaries on the finite deformation field and notations, and all relevant expressions in the deformed state.

[1]Total expression is in terms of its components and the unit vectors, for instance, a second-order tensor $\boldsymbol{A} = A_{ij}\boldsymbol{e}^i\boldsymbol{e}^j = A^i_j\boldsymbol{e}_i\boldsymbol{e}_j = A^{ij}\boldsymbol{e}_i\boldsymbol{e}_j$.

2 Formulations in undeformed state using deformation gradient

In this section, we will follow Route 1 shown in Fig. 2 to find the explicit expression of $\Delta\boldsymbol{u}$ in terms of the Riemann curvature tensor $\boldsymbol{R}$ by using the deformation gradient $\boldsymbol{F}$.

$$\Delta\boldsymbol{u} \xrightarrow[\oint(\boldsymbol{F}-\boldsymbol{I})\cdot\mathrm{d}\boldsymbol{X}]{\text{Route 1}} \boldsymbol{R}$$

Fig. 2 Route 1 using deformation gradient $\boldsymbol{F}$

2.1 Displacement vector variation

Definition 1 *Let $\Delta\boldsymbol{u}$ be the change of the displacement vector along a closed curve. The definition of the displacement vector variation $\Delta\boldsymbol{u}$ is given by*

$$\begin{aligned}\Delta\boldsymbol{u} &= \oint_{\partial\Psi}\mathrm{d}\boldsymbol{u} = \oint_{\partial\Psi}\mathrm{d}(\boldsymbol{x}-\boldsymbol{X}) = \oint_{\partial\Psi}\frac{\partial(\boldsymbol{x}-\boldsymbol{X})}{\partial\boldsymbol{X}}\cdot\mathrm{d}\boldsymbol{X}\\ &= \oint_{\partial\Psi}\Big(\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}}-\frac{\partial\boldsymbol{X}}{\partial\boldsymbol{X}}\Big)\cdot\mathrm{d}\boldsymbol{X} = \oint_{\partial\Psi}(\boldsymbol{F}-\boldsymbol{I})\cdot\mathrm{d}\boldsymbol{X},\end{aligned} \tag{1}$$

where the closed loop $\partial\Psi$ is the boundary of the surface domain Ψ.

Theorem 1 *Let* $\boldsymbol{u}$ and $\boldsymbol{R}$ *be the displacement field and the Riemann curvature tensor, respectively. In the undeformed state $\mathfrak{B}$, the displacement vector variation $\Delta\boldsymbol{u}$ can be presented explicitly in terms of the Riemann curvature tensor as follows*:

$$\begin{aligned}\Delta\boldsymbol{u} &= -\frac{1}{2}\iint_{\Psi}(\boldsymbol{\varepsilon}:\boldsymbol{R}(\boldsymbol{G}_A,\boldsymbol{G}_B)\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{A}\\ &= -\frac{1}{2}\iint_{\Psi}(\boldsymbol{u}\boldsymbol{R}(\boldsymbol{G}_A,\boldsymbol{G}_B):\boldsymbol{\varepsilon})\cdot\mathrm{d}\boldsymbol{A},\end{aligned} \tag{2}$$

where $\boldsymbol{\varepsilon}$ is the permutation tensor.

Proof Since $\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{u}$, Eq. (1) can be rewritten as

$$\Delta\boldsymbol{u} = \oint_{\partial\Psi}\mathrm{d}\boldsymbol{u} = \oint_{\partial\Psi}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{X}}\cdot\mathrm{d}\boldsymbol{X} = \oint_{\partial\Psi}(\boldsymbol{F}-\boldsymbol{I})\cdot\mathrm{d}\boldsymbol{X}, \tag{3}$$

where $\boldsymbol{F}=\boldsymbol{I}+\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{X}}=\boldsymbol{I}+\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{X}}$, where $\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{X}}$ is the displacement gradient with respect to $\boldsymbol{X}$, and $\boldsymbol{F}-\boldsymbol{I}=(\nabla_A\boldsymbol{u})\boldsymbol{G}^A$. The displacement change can be expressed as

$$\Delta\boldsymbol{u} = \oint_{\partial\Psi}(\nabla_A\boldsymbol{u})\boldsymbol{G}^A\cdot\mathrm{d}\boldsymbol{X}. \tag{4}$$

In terms of the Stokes theorem, Eq. (4) can be transferred into the surface integration as follows:

$$\begin{aligned}\Delta\boldsymbol{u} &= -\iint_{\Psi}(\boldsymbol{F}-\boldsymbol{I})\times\boldsymbol{\nabla}_{\boldsymbol{X}}\cdot\mathrm{d}\boldsymbol{A}\\ &= -\iint_{\Psi}(\boldsymbol{F}\times\boldsymbol{\nabla}_{\boldsymbol{X}}-\underbrace{\boldsymbol{I}\times\boldsymbol{\nabla}_{\boldsymbol{X}}}_{=0})\cdot\mathrm{d}\boldsymbol{A}\\ &= -\iint_{\Psi}\boldsymbol{F}\times\boldsymbol{\nabla}_{\boldsymbol{X}}\cdot\mathrm{d}\boldsymbol{A}\\ &= -\iint_{\Psi}(\nabla_A\boldsymbol{u})\boldsymbol{G}^A\times\boldsymbol{\nabla}_{\boldsymbol{X}}\cdot\mathrm{d}\boldsymbol{A},\end{aligned} \tag{5}$$

in which $\boldsymbol{I}$ is a unit symmetric tensor, the identity $\boldsymbol{I} \times \boldsymbol{\nabla} = \boldsymbol{0}$ is used, and $\mathrm{d}\boldsymbol{A} = \mathrm{d}\boldsymbol{X} \times \mathrm{d}\boldsymbol{X}$ is the surface element vector in the undeformed state $\mathfrak{B}$.

Since the gradient operator $\boldsymbol{\nabla}_{\boldsymbol{X}} = \frac{\partial}{\partial X^B}\boldsymbol{G}^B = \nabla_B \boldsymbol{G}^B$, we can continue with the calculation of Eq. (5),

$$\begin{aligned}\Delta\boldsymbol{u} &= -\iint_\Psi (\nabla_A \boldsymbol{u})\boldsymbol{G}^A \times (\boldsymbol{G}^B \nabla_B) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\iint_\Psi ((\nabla_A \nabla_B \boldsymbol{u})\boldsymbol{G}^A \times \boldsymbol{G}^B) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\iint_\Psi ((\nabla_A \nabla_B \boldsymbol{u})\varepsilon^{ABC}\boldsymbol{G}^C) \cdot \mathrm{d}\boldsymbol{A},\end{aligned} \tag{6}$$

where ε^{ABC} is the permutation symbol. Using the property of ε^{ABC}, the sign of ε^{ABC} will be changed if we change the order of A and B, namely, $\varepsilon^{ABC} = -\varepsilon^{BAC}$. Therefore, Eq. (6) can be rewritten as

$$\begin{aligned}\Delta\boldsymbol{u} &= -\frac{1}{2}\iint_\Psi ((\nabla_A \nabla_B \boldsymbol{u})\varepsilon^{ABC}\boldsymbol{G}^C + (\nabla_A \nabla_B \boldsymbol{u})\varepsilon^{ABC}\boldsymbol{G}^C) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_\Psi ((\nabla_A \nabla_B \boldsymbol{u})\varepsilon^{ABC}\boldsymbol{G}^C - (\nabla_B \nabla_A \boldsymbol{u})\varepsilon^{ABC}\boldsymbol{G}^C) \cdot \mathrm{d}\boldsymbol{A} \\ &= \frac{1}{2}\iint_\Psi ((\nabla_B \nabla_A \boldsymbol{u} - \nabla_A \nabla_B \boldsymbol{u})\varepsilon^{ABC}\boldsymbol{G}^C) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_\Psi ((\nabla_A \nabla_B \boldsymbol{u} - \nabla_B \nabla_A \boldsymbol{u})\varepsilon^{ABC}\boldsymbol{G}^C) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_\Psi ((\nabla_A \nabla_B \boldsymbol{u} - \nabla_B \nabla_A \boldsymbol{u})\boldsymbol{G}^A \times \boldsymbol{G}^B) \cdot \mathrm{d}\boldsymbol{A}.\end{aligned} \tag{7}$$

Introducing the Riemann curvature tensor[2], for Eq. (7), we have the Riemann curvature tensor component $\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)$

$$\begin{aligned}\nabla_A \nabla_B \boldsymbol{u} - \nabla_B \nabla_A \boldsymbol{u} &= (\nabla_A, \nabla_B)\boldsymbol{u} \\ &= \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u}.\end{aligned} \tag{8}$$

When we substitute Eq. (8) and $\boldsymbol{G}^A \times \boldsymbol{G}^B = \boldsymbol{G}^A\boldsymbol{G}^B : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} : \boldsymbol{G}^A\boldsymbol{G}^B$ into Eq. (7), we obtain

$$\begin{aligned}\Delta\boldsymbol{u} &= -\frac{1}{2}\iint_\Psi (\boldsymbol{u}\boldsymbol{R}(\boldsymbol{X}_A, \boldsymbol{X}_B)\boldsymbol{G}^A \times \boldsymbol{G}^B) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_\Psi (\boldsymbol{u}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{G}^A\boldsymbol{G}^B : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_\Psi (\boldsymbol{\varepsilon} : \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{G}^A\boldsymbol{G}^B\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A},\end{aligned} \tag{9}$$

where the permutation tensor $\boldsymbol{\varepsilon} = \varepsilon_{ABC}\boldsymbol{G}^A\boldsymbol{G}^B\boldsymbol{G}^C$.

[2]The Riemann tensor is given in terms of the Levi-Civita connection ∇ by the following formula: $\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}$, where $[\boldsymbol{u},\boldsymbol{v}]$ is the Lie bracket of vector fields. For each pair of tangent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})$ is a linear transformation of the tangent space of the manifold. It is linear in $\boldsymbol{u}$ and $\boldsymbol{v}$, and thus defines a tensor. If $\boldsymbol{u} = \frac{\partial}{\partial \boldsymbol{x}^i}$ and $\boldsymbol{v} = \frac{\partial}{\partial \boldsymbol{x}^j}$ are coordinate vector fields, then $[\boldsymbol{u},\boldsymbol{v}] = \boldsymbol{0}$, and therefore the formula simplifies to $\boldsymbol{R}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w}$. The curvature tensor measures noncommutativity of the covariant derivative, and as such is the integrability obstruction for the existence of an isometry with Euclidean space (called, in this context, flat space). The linear transformation $\boldsymbol{w} \mapsto R(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w}$ is also called the curvature transformation or endomorphism.

Introducing the Riemann curvature tensor $\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B) = \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{G}^A\boldsymbol{G}^B$, the relationship between the displacement vector variation and the Riemann tensor can be finally established as follows:

$$\begin{aligned}\Delta\boldsymbol{u} &= -\frac{1}{2}\iint_\Psi (\boldsymbol{\varepsilon} : \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_\Psi (\boldsymbol{u}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B) : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A}. \end{aligned} \tag{10}$$

This explicit relationship between $\Delta\boldsymbol{u}$ and Riemann tensor $\boldsymbol{R}$ states that the displacement vector variation generally does not vanish around a closed loop.

With the explicit relationship (10), the compatibility condition can be stated as that the vanishing of displacement vector variation will lead to the vanishing of the Riemann tensor. The proof is complete.

The above results can be expressed in the conventional format. For example, Eq. (8) can be represented as

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} &= \nabla_A\nabla_B\boldsymbol{u} - \nabla_B\nabla_A\boldsymbol{u} \\ &= (\nabla_A\nabla_B u_K - \nabla_B\nabla_A u_K)\boldsymbol{G}^K \\ &= R^J_{.KAB} u_J \boldsymbol{G}^K \\ &= R^J_{.KAB}\boldsymbol{u} \cdot \boldsymbol{G}_J\boldsymbol{G}^K. \end{aligned} \tag{11}$$

Thus,

$$\begin{aligned}\Delta\boldsymbol{u} &= -\frac{1}{2}\iint_\Psi (R^J_{.KAB}\boldsymbol{u} \cdot \boldsymbol{G}_J\boldsymbol{G}^K\boldsymbol{G}^A\boldsymbol{G}^B : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_\Psi (\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A}, \end{aligned} \tag{12}$$

where the Riemann curvature tensor $\boldsymbol{R} = R^J_{.KAB}\boldsymbol{G}_J\boldsymbol{G}^K\boldsymbol{G}^A\boldsymbol{G}^B$, and $R^J_{.KAB}$ are the components of the Riemann tensor[33,42],

$$R^J_{.KAB} := \partial_A\Gamma^J_{BK} - \partial_B\Gamma^J_{AK} + \Gamma^J_{AM}\Gamma^M_{BK} - \Gamma^J_{BM}\Gamma^M_{AK}, \tag{13}$$

where the Christoffel symbols Γ^I_{JK} are called the coefficients of the affine connections, or Christoffel symbols, with respect to the frame $\boldsymbol{G}_J$, that is to say, $\boldsymbol{\nabla}_{\boldsymbol{G}_J}\boldsymbol{G}_K = \boldsymbol{G}_I\Gamma^I_{JK}$.

2.2 Displacement flux tensor

Theorem 2 *Let* $\boldsymbol{u}$ and $\boldsymbol{R}$ *be the displacement field and the Riemann curvature tensor, respectively. The curl of the deformation gradient* curl $\boldsymbol{F}$ *can be expressed explicitly in terms of the Riemann curvature tensor as follows*:

$$\operatorname{curl}\boldsymbol{F} = \boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}, \tag{14}$$

or in the conventional form:

$$\operatorname{curl}\boldsymbol{F} = \frac{1}{2}R^J_{.KAB}\boldsymbol{u} \cdot \boldsymbol{G}_J\boldsymbol{G}^K\boldsymbol{G}^A\boldsymbol{G}^B : \boldsymbol{\varepsilon} = \frac{1}{2}R^J_{.KAB}u_J \cdot \boldsymbol{G}^K\boldsymbol{G}^A \times \boldsymbol{G}^B. \tag{15}$$

Proof From the previous formulations (5) and (12), we have $\Delta\boldsymbol{u} = \oint_{\partial\Psi}\mathrm{d}\boldsymbol{u} = \oint_{\partial\Psi}\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{X} = \oint_{\partial\Psi}(\boldsymbol{F} - \boldsymbol{I}) \cdot \mathrm{d}\boldsymbol{X} = -\iint_\Psi \boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{A} = -\iint_\Psi \boldsymbol{u} \cdot \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B) : \boldsymbol{\varepsilon} \cdot \mathrm{d}\boldsymbol{A}$. Then, we get (14) and (15). The proof is complete.

Since $\boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{F}\boldsymbol{\nabla}_{\boldsymbol{X}} : \boldsymbol{\varepsilon}$, the relationship between $\boldsymbol{F}$ and the Riemann tensor $\boldsymbol{R}$ can be also expressed as follows:

$$\boldsymbol{F}\boldsymbol{\nabla}_{\boldsymbol{X}} = \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R}. \tag{16}$$

Since the curvature transformation or endomorphism is linear, the 2nd derivatives of the displacement vector $\boldsymbol{u}$ in the $\boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}}$ have been transferred into a linear form of $\boldsymbol{u}$. The beauty is that there are no derivatives of the displacement vector on the right hand of Eq. (14). Furthermore, since the geometric meaning of the Riemann curvature $\boldsymbol{R}$ represents the curvature of space, Eqs. (14) and (15) clearly reveal the geometrical nature of $\boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}}$, namely, the curl of the deformation gradient $\boldsymbol{F} \times \boldsymbol{\nabla}_{\boldsymbol{X}}$ is linearly proportional to the Riemann curvature $\boldsymbol{R}$ and the displacement vector $\boldsymbol{u}$.

Due to the arbitrary nature of the displacement vector $\boldsymbol{u}$, the compatibility condition for a simply-connected body can be stated that the compatibility condition curl $\boldsymbol{F} = 0$ is equivalent to the vanishing of the Riemann tensor as $\boldsymbol{R} = \boldsymbol{0}$.

Since the surface Ψ and its closed boundary $\partial\Psi$ are arbitrary, for infinitesimal contours $\partial\Psi$, we get from Eq. (10)

$$\begin{aligned} \mathrm{d}\Delta\boldsymbol{u} &= -\frac{1}{2}(\boldsymbol{\varepsilon} : \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}(\boldsymbol{u}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B) : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \boldsymbol{T} = \frac{\mathrm{d}\Delta\boldsymbol{u}}{\mathrm{d}\boldsymbol{A}} &= -\frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} \\ &= -\frac{1}{2}\boldsymbol{u}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B) : \boldsymbol{\varepsilon} \\ &= -\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}. \end{aligned} \tag{18}$$

$\boldsymbol{T} = \frac{\mathrm{d}\Delta\boldsymbol{u}}{\mathrm{d}\boldsymbol{A}}$ is called the displacement flux tensor defined in the undeformed state $\mathfrak{B}$. The displacement density tensor is related to the continua dislocation density tensor[8–16,18–22]. (18) can also be rewritten as $\boldsymbol{\varepsilon} \cdot \frac{\mathrm{d}\Delta\boldsymbol{u}}{\mathrm{d}\boldsymbol{A}} = -\boldsymbol{u} \cdot \boldsymbol{R}$.

From the above formulations, we can have a corollary as follows.

Corollary 1 *The symmetric part of the deformation gradient $\boldsymbol{F}$ has no contribution to the displacement change $\Delta\boldsymbol{u}$ and the displacement density tensor $\boldsymbol{T}$.*

Proof Let $\boldsymbol{S} = \frac{1}{2}(\boldsymbol{F}+\boldsymbol{F}^{\mathrm{T}})$ and $\boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{F}-\boldsymbol{F}^{\mathrm{T}})$ be the symmetric part and anti-symmetric part of the deformation gradient $\boldsymbol{F}$, respectively. As we know, any tensor can be decomposed into a symmetric and an antisymmetric part, namely, $\boldsymbol{F} = \boldsymbol{S} + \boldsymbol{\Omega}$. Since the curl of the symmetric tensor vanishes curl $\boldsymbol{S} = \boldsymbol{S} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{0}$, curl $\boldsymbol{F} =$ curl $(\boldsymbol{S} + \boldsymbol{\Omega}) =$ curl $\boldsymbol{\Omega} = \boldsymbol{\Omega} \times \boldsymbol{\nabla}_{\boldsymbol{X}}$. Hence, we have

$$\begin{aligned} \operatorname{curl} \boldsymbol{\Omega} &= \operatorname{curl}\left(\frac{1}{2}(\boldsymbol{F} - \boldsymbol{F}^{\mathrm{T}})\right) \\ &= \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} \\ &= \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}. \end{aligned} \tag{19}$$

curl $\boldsymbol{S} = \boldsymbol{0}$ indicates that the symmetric part of the deformation gradient $\boldsymbol{F}$ has no contribution to the comparability conditions. In other words, the symmetric deformations are always compatible, and the incompatible deformation will make the symmetric deformation break down.

Since curl $\boldsymbol{\Omega} = \boldsymbol{\Omega} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{\Omega}\boldsymbol{\nabla}_{\boldsymbol{X}} : \boldsymbol{\varepsilon}$, $\boldsymbol{\Omega}\boldsymbol{\nabla}_{\boldsymbol{X}} : \boldsymbol{\varepsilon} = \boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} : \boldsymbol{\varepsilon}$, we have

$$\boldsymbol{\Omega}\boldsymbol{\nabla}_{\boldsymbol{X}} = \frac{1}{2}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} = \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R}. \tag{20}$$

2.3 Curl of rate of deformation tensor

In continuum mechanics, the strain rate tensor $\boldsymbol{D}$ or $\boldsymbol{d}$ is a physical quantity that describes the rate of deformation change of a material in the neighbourhood of a certain point, and at a certain moment in time. It can be defined as the derivative of the strain tensor with respect to the time, or as the symmetric component of the gradient of the flow velocity $\dot{\boldsymbol{u}}$, namely, the rate of the deformation tensor $\boldsymbol{L}$ or $\boldsymbol{l}$.

The strain rate tensor is a purely kinematic concept that describes the macroscopic motion of the material. Therefore, it does not depend on the nature of the material, or on the forces and stresses that may be acting on it, and it can be applied to any continuous medium, whether solid, liquid or gas.

In the undeformed state $\mathfrak{B}$, taking a material derivative to Eq. (1) with respect to time

$$\begin{aligned}\Delta\dot{\boldsymbol{u}} &= \oint_{\partial\Psi} \mathrm{d}\dot{\boldsymbol{u}} = \oint_{\partial\Psi} \dot{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{x} = \oint_{\partial\Psi} \boldsymbol{l} \cdot \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} \\ &= \oint_{\partial\Psi} \dot{\boldsymbol{F}} \cdot \mathrm{d}\boldsymbol{X} = -\iint_{\Psi} \dot{\boldsymbol{F}} \times \nabla_{\boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{A} \\ &= -\frac{1}{2}\iint_{\Psi} (\dot{\boldsymbol{u}} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A}. \end{aligned} \tag{21}$$

Hence, the explicit relationship between $\dot{\boldsymbol{F}}$ and the Riemann tensor $\boldsymbol{R}$ is

$$\dot{\boldsymbol{F}} \times \nabla_{\boldsymbol{X}} = \frac{1}{2}\dot{\boldsymbol{u}} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}. \tag{22}$$

Since $\dot{\boldsymbol{F}} \times \nabla_{\boldsymbol{X}} = \boldsymbol{F}\nabla_{\boldsymbol{X}} : \boldsymbol{\varepsilon}$,

$$\dot{\boldsymbol{F}}\nabla_{\boldsymbol{X}} = \frac{1}{2}\dot{\boldsymbol{u}} \cdot \boldsymbol{R}. \tag{23}$$

All formulations in the undeformed state are summarized in Table 1.

Table 1 Total tensorial expressions

Parameter	Undeformed state	Deformed state
$\Delta\boldsymbol{u}$	$-\frac{1}{2}\iint_{\Psi}(\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}) \cdot \mathrm{d}\boldsymbol{A}$	$-\frac{1}{2}\iint_{\Psi} \boldsymbol{u} \cdot \boldsymbol{r} : \boldsymbol{\varepsilon} \cdot \mathrm{d}\boldsymbol{a}$
$\boldsymbol{F} \times \boldsymbol{\nabla}$	$\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}$	$\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{r} : \boldsymbol{\varepsilon}$
$\boldsymbol{F}\boldsymbol{\nabla}$	$\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R}$	$\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{r}$
$\boldsymbol{T}$	$-\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R} : \boldsymbol{\varepsilon}$	$-\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{r} : \boldsymbol{\varepsilon}$
$\boldsymbol{\Omega}\boldsymbol{\nabla}$	$\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R}$	$\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{r}$
$\boldsymbol{\varepsilon} \cdot (\boldsymbol{\omega}\boldsymbol{\nabla})$	$-\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R}$	$-\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{r}$
$\dot{\boldsymbol{F}}\nabla$	$\frac{1}{2}\dot{\boldsymbol{u}} \cdot \boldsymbol{R}$	$\frac{1}{2}\dot{\boldsymbol{u}} \cdot \boldsymbol{r}$
$\boldsymbol{l}\nabla$	$\frac{1}{2}\dot{\boldsymbol{u}} \cdot \boldsymbol{R}$	$\frac{1}{2}\dot{\boldsymbol{u}} \cdot \boldsymbol{r}$

Corresponding formulations in the deformed state $\mathfrak{b}$ are given in Appendix C.

3 Formulations by exterior differential forms using deformation gradient

The previous results are still dependent on the choice of coordinates. In order to generalize them into the coordinate-free form, let us reformulate them by using exterior differential forms. Exterior differential forms arise when concepts as the work of a field along a path and flux of a fluid through a surface are generalized to higher dimensions[44–45], which provides a unified approach to defining integrands over curves, surfaces, volumes, and higher-dimensional manifolds. The modern notion of differential forms was pioneered by Cartan[46–47], who made it possible to extend this algebraic structure to include the exterior differential forms by employing exterior products of differentials of coordinates. It was then possible to define exterior differential form fields on differentiable manifolds that are locally equivalent to Euclidean spaces and to introduce an analysis of forms in which only the first order derivatives survive. It was soon realised that this analysis would be one of the most powerful, perhaps indispensable tools of the modern differential geometry, and many mathematical properties could be relatively easily revealed by almost algebraic operations. Moreover, it is perhaps not wrong to claim that the mathematical structure of theoretical physics today is entirely based on the formalism of differential geometry. We also observe that this formalism is increasingly infiltrating into engineering sciences to study some fundamental problems and even in many practical applications. Therefore, the exterior analysis is no longer in the realm of mathematicians. It seems that it would now be quite beneficial for physicist and engineers to acquire a rather good skill in dealing with exterior forms[41,45,49–50]. In this section, we will take some materials from those books and apply to our problem. In those mathematics books, all formulations are presented in the component form. Therefore, we have to convert all formulations into total tensor expressions, which is an innovative point of this paper.

Theorem 3 *In the undeformed state* $\mathfrak{B}$, *let* $\boldsymbol{u}$ *be the displacement field and* $\boldsymbol{R}$ *be the Riemann curvature tensor. The displacement change* $\Delta\boldsymbol{u}$, $\mathrm{d}\boldsymbol{F}$ *and the displacement density tensor* $\boldsymbol{T}$ *can be expressed explicitly in terms of the Riemann curvature tensor* $\boldsymbol{R}$ *as follows*:

$$\Delta\boldsymbol{u} = -\frac{1}{2}\int_{\Psi}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{A}, \tag{24}$$

$$\mathrm{d}\boldsymbol{F} = -\frac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}, \tag{25}$$

$$\boldsymbol{T} = -\frac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}. \tag{26}$$

Proof Let $\mathfrak{B}$ be the undeformed state and $\mathrm{d}\boldsymbol{u} = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{Y} = \boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{Y}$, $(\boldsymbol{X},\boldsymbol{Y})\in\mathfrak{B}$. According to the Stokes integration theorem, we have the displacement vector variation along an arbitrary closed loop as follows:

$$\begin{aligned}\Delta\boldsymbol{u} &= \oint_{\partial\Psi}\mathrm{d}\boldsymbol{u} = \int_{\Psi}\mathrm{d}(\mathrm{d}\boldsymbol{u})\\ &= \int_{\Psi}\mathrm{d}(\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{Y}) = \int_{\Psi}\mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{Y}),\end{aligned} \tag{27}$$

in which $\partial\Psi$ is the closed boundary of a surface $\Psi\in\mathfrak{B}$, and $\mathrm{d}\boldsymbol{u} = \boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{Y}$ is a vector-valued 1-form. Differentiating the above equation once more yields the vector-valued 2-form,

$$\begin{aligned}\mathrm{d}(\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{Y}) &= \mathrm{d}\boldsymbol{F}\wedge\mathrm{d}\boldsymbol{Y} + (-1)^0\boldsymbol{F}\wedge\mathrm{d}^2\boldsymbol{Y}\\ &= \mathrm{d}\boldsymbol{F}\wedge\mathrm{d}\boldsymbol{Y}\\ &= \mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{Y})\\ &= \mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u})\wedge\mathrm{d}\boldsymbol{Y} + (-1)^0\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\mathrm{d}^2\boldsymbol{Y}\\ &= \boldsymbol{\nabla}_{\boldsymbol{Y}}(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}\\ &= \boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y},\end{aligned} \tag{28}$$

where Poincaré Lemma[48] is used for $\mathrm{d}^2\boldsymbol{Y}=\boldsymbol{0}$.

Due to the antisymmetric nature of exterior algebra, $\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}=-\mathrm{d}\boldsymbol{Y}\wedge\mathrm{d}\boldsymbol{X}$,

$$\mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{Y})=-\frac{1}{2}(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}. \tag{29}$$

According to the definition of the Riemann operator

$$\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}=\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u}-\boldsymbol{\nabla}_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{u}, \tag{30}$$

and in the coordinate frame, the torsion curvature $\boldsymbol{\nabla}_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{u}=\boldsymbol{0}$, we have

$$\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u}=\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}. \tag{31}$$

If we expand the vector fields in terms of the coordinate basis ∂_I, the Riemann tensor $\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}$ $=(R^I_{JKL}X^KY^Lu^J)\partial_I$ and its components $R^I_{JKL}:=\partial_K\Gamma^I_{LJ}-\partial_L\Gamma^I_{KJ}+\Gamma^I_{KM}\Gamma^M_{LJ}-\Gamma^I_{LM}\Gamma^M_{KJ}$. The symbol Γ^I_{JK} is called the coefficients of the affine connections, or the Christoffel symbols, with respect to the frame $\boldsymbol{G}_J$, that is, $\boldsymbol{\nabla}_{\boldsymbol{G}_J}\boldsymbol{G}_K=\boldsymbol{G}_I\Gamma^I_{JK}$.

Therefore, we have

$$\begin{aligned}\mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{Y})&=-\frac{1}{2}(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}\\&=-\frac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}.\end{aligned} \tag{32}$$

Finally, we have the displacement change in differential forms

$$\begin{aligned}\Delta\boldsymbol{u}&=-\int_\Psi\mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{Y})\\&=-\frac{1}{2}\int_\Psi(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}\\&=-\frac{1}{2}\int_\Psi\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}.\end{aligned} \tag{33}$$

Note that the area element $\mathrm{d}\boldsymbol{A}=\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}$, hence, the dislocation density tensor $\boldsymbol{T}$ and the incompatibility operator $\mathrm{inc}(\boldsymbol{F})$

$$\boldsymbol{T}=\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\boldsymbol{A}}=-\frac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}. \tag{34}$$

Since $\mathrm{d}\boldsymbol{F}\wedge\mathrm{d}\boldsymbol{Y}=-(1/2)\boldsymbol{R}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}$, we have

$$\mathrm{d}\boldsymbol{F}=\boldsymbol{T}\cdot\mathrm{d}\boldsymbol{X}=-\frac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}. \tag{35}$$

The proof is complete.

Note that the above $\boldsymbol{F}$ is not unique. In fact, if $\boldsymbol{\Pi}=\boldsymbol{F}+\mathrm{d}\boldsymbol{\Lambda}$ for any forms $\boldsymbol{\Lambda}$ and $\boldsymbol{\Pi}$, because $\mathrm{d}(\mathrm{d}\boldsymbol{\Lambda})$ is identical to zero,

$$\mathrm{d}\boldsymbol{\Pi}=\mathrm{d}(\boldsymbol{F}+\mathrm{d}\boldsymbol{\Lambda})=\mathrm{d}\boldsymbol{F}=-\frac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}. \tag{36}$$

This freedom of choice in selecting $\boldsymbol{\Pi}$ is called the gauge invariance, and its generalization plays an important role in physics.

Making a material time derivative on the above equation renders the rate of the deformation gradient,

$$\mathrm{d}\dot{\boldsymbol{F}}=-\frac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\dot{\boldsymbol{u}}\cdot\mathrm{d}\boldsymbol{X}. \tag{37}$$

All formulations are listed in Table 2.

Table 2 Exterior differential forms

Parameter	Undeformed state	Deformed state
$\Delta\boldsymbol{u}$	$-\dfrac{1}{2}\iint_{\Psi}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}\wedge\mathrm{d}\boldsymbol{Y}$	$-\dfrac{1}{2}\iint_{\Psi}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y}$
$\mathrm{d}\boldsymbol{F}$	$-\dfrac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{X}$	$-\dfrac{1}{2}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}$
$\mathrm{d}\dot{\boldsymbol{F}}$	$-\dfrac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\dot{\boldsymbol{u}}\cdot\mathrm{d}\boldsymbol{X}$	$-\dfrac{1}{2}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\dot{\boldsymbol{u}}\cdot\mathrm{d}\boldsymbol{x}$
$\boldsymbol{T}$	$-\dfrac{1}{2}\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{u}$	$-\dfrac{1}{2}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}$

Using the language of differential forms, the compatibility conditions in the deformed state $\mathfrak{B}$ can be simply stated as follows:

(i) Compatibility conditions of simply-connected bodies

For simply-connected bodies, the displacement change integral along the closed loop must be zero, $\mathrm{d}(\boldsymbol{F})=\boldsymbol{0}$. Therefore, the Riemann tensor $\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})$ must vanish because of the arbitrary nature of the displacement $\boldsymbol{u}$.

(ii) Compatibility conditions of non-simply-connected bodies

For non-simply-connected bodies, the vanishing of Riemann tensor is not enough and needs extra conditions. According to the theorem of de Rham[47,51–52], these extra conditions are $\oint_{\partial\Psi_i}\mathrm{d}\boldsymbol{u}=\oint_{\partial\Psi_i}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X}=\boldsymbol{0}$, where $\partial\Psi_i$ are the closed loops, including the holes and/or defects, which has been well studied by Yavari in 2013[40].

4 Burgers vector and dislocation density tensor

In materials science, a dislocation is a crystallographic defect, or an irregularity within a crystal structure. The presence of dislocations strongly influences most of the materials properties. The theory that describes the elastic fields of the defects was originally developed by Volterra[30] in 1907, but the term “dislocation”, which refers to a defect on the atomic scale was coined by Taylor[5] in 1934 to explain the plastic deformation of a single crystal. Mathematically, dislocations are a type of topological defect and in some way the incompatibility is related to the dislocation.

Nye[8] studied the geometry of dislocation under small deformations. The idea that the geometry of a dislocated crystal can be appropriately represented in terms of non-Euclidean space was first introduced by Kondo[9–10] and Bilby et al.[11], independently, reaching its culmination in Kröner’s essay[12]. The study of dislocation geometry shows that the space in which the dislocated continuum is imbedded will have a non-zero Riemann curvature, therefore, the space is non-Euclidean but the Riemannian space.

In order to compare with the well-known results of Kröner[12], and Le and Stumpf[18–19], let us apply the previous formulations to the Burgers vector. In physics, the Burgers vector is a vector, which is often denoted as $\boldsymbol{b}$, and represents the magnitude and direction of the lattice distortion that results from a dislocation in a crystal lattice.

Given a material with a distribution of dislocations, the Burger vector $\boldsymbol{b}$ enclosed by a curve $\partial\Psi$ in the reference state is commonly defined in terms of an integral,

$$\boldsymbol{b}=\oint_{\partial\Psi}\mathrm{d}\boldsymbol{u}. \tag{38}$$

When one compares this definition with the displacement change that is integral in the profuse section, the Burgers vector $\boldsymbol{b}$ can definitely be expressed in the Riemann tensor,

$$\boldsymbol{b}=\oint_{\partial\Psi}\mathrm{d}\boldsymbol{u}=-\frac{1}{2}\int_{\Psi}(\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon})\cdot\mathrm{d}\boldsymbol{A}=-\frac{1}{2}\int_{\Psi}\boldsymbol{u}\cdot\boldsymbol{r}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{a}. \tag{39}$$

For infinitesimal contours $\partial\Psi$, the above equation can be expressed as follows:

$$\mathrm{d}\boldsymbol{b} = -\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{A} = -\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{r}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{a}. \tag{40}$$

Since the dislocation $\boldsymbol{b}\neq\boldsymbol{0}$, the Riemann tensors in both undeformed and deformed states do not vanish, that is, $\boldsymbol{R}\neq\boldsymbol{0}$ and $\boldsymbol{r}\neq\boldsymbol{0}$.

If we compare Eqs. (39) and (40) with Eqs. (56) and (57) of Ref. [12], it is easy to see that they are completely the same except in different notations. Here, we use total expressions of tensor and the component form used by Kröner, who obtained the Burgers vector component $b^k=\frac{1}{2}\iint(\partial_m A_l^k-\partial_l A_m^k)\mathrm{d}F^{ml}$ and its variation $\Delta b^k=\frac{1}{2}(\partial_m A_l^k-\partial_l A_m^k)\Delta F^{ml}$, where A_l^k is the deformation gradient, and ΔF^{ml} is the small surface element[12].

From Eq. (39) we can define the dislocation density tensors $\boldsymbol{T}$ and $\boldsymbol{t}$ for the undeformed and deformed states, respectively,

$$\begin{cases}\boldsymbol{T}=\dfrac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}\boldsymbol{A}}=-\dfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon}, & \text{undeformed state,}\\ \boldsymbol{t}=\dfrac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}\boldsymbol{a}}=-\dfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{r}:\boldsymbol{\varepsilon}, & \text{deformed state.}\end{cases} \tag{41}$$

It is clear that both dislocation density tensors are explicitly linked to the Riemann tensors. The basic formula is listed in the table below.

Table 3 Basic dislocation formula

Parameter	Undeformed state	Deformed state
$\boldsymbol{b}$	$-\frac{1}{2}\int_\Psi(\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon})\cdot\mathrm{d}\boldsymbol{A}$	$-\frac{1}{2}\int_\Psi\boldsymbol{u}\cdot\boldsymbol{r}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{a}$
$\boldsymbol{T}$	$-\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon}$	$-\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{r}:\boldsymbol{\varepsilon}$

If we compare the second equation of (41) with Eq. (59) of Kröner[12], it is also easy to see that they have completely the same format except in different notations. Kröner obtained the Burgers vector component variation $\Delta b^k\equiv\alpha_{ml}^k\Delta F^{ml}$, where $\alpha_{ml}^k=\frac{1}{2}A_\kappa^k(\partial_m A_l^\kappa-\partial_l A_m^\kappa)$ is dislocation density tensor components in the deformed state[12].

In 1994 and 1995, Le and Stumpf[18–19] obtained the similar results while they studied the finite elasoplaticity with microstructures, they derived the Burgers vector component in the undeformed state as $(\boldsymbol{b})^a=\frac{1}{2}((\boldsymbol{F}^{\mathrm{e}-1})_{c,b}^\alpha-(\boldsymbol{F}^{\mathrm{e}-1})_{b,c}^\alpha)\mathrm{d}x^b\wedge\mathrm{d}x^c$, where $\boldsymbol{F}^{\mathrm{e}}$ is the elastic deformation gradient[18]. Since the elastic deformation gradient $\boldsymbol{F}^{\mathrm{e}}$ cannot be the gradient of global maps $\boldsymbol{u}$ ($\boldsymbol{F}^{\mathrm{e}}$ is incompatible)[6], the above formula cannot be further expressed in terms of the Riemann tensor. Nevertheless, the formulation of Le and Stumpf[18] is very general, and in the case of deformation without plastic, their result can be reduced to Eqs. (39), (40), and (41) of this study. From this point of view, Eqs. (39), (40), and (41) will be consistent with the theory of Le and Stumpf[18–19] when the plastic deformation is not taken into account.

5 Displacement vector variation derived without using deformation gradient

After completing the previous formulations along Route 1, we find that Eqs. (12) and (33) taking the same form of a change on a vector is parallel displaced around a closed curve (Page 262, Frankel[41]), the only difference is that the formulation process of Eq. (12) does not require the displacement vector $\boldsymbol{u}$ being parallel transported around a closed loop $\partial\Psi$ and is purely based on the deformation gradient without any other manipulation.

We then notice that, in 1960, Kröner[12] took the vector difference $\Delta C^k = -\frac{1}{2} R^k_{nml} C^l \Delta F^{nm}$ from differential geometry textbooks and applied to his dislocation theory without giving a justification, although the equation on the vector C^k is also derived by using the concept of parallel displaced vector.

Now we have a question, i.e., are Eqs. (12) and (33) derived from the deformation gradient exactly the same as the C^k formulated from the pure geometric perspective? We believe that the answering of this question is important to know the validation scope of the mathematical formula and in the same time provides an insurance on its applications.

To answer the question, we will follow Route 2 in Fig. 3 to find the explicit expression of $\Delta \boldsymbol{u}$ in terms of the Riemann curvature tensor $\boldsymbol{R}$ without using the deformation gradient $\boldsymbol{F}$.

$$\Delta \boldsymbol{u} \xrightarrow[\oint \mathrm{d}\boldsymbol{u}]{\text{Route 2}} \boldsymbol{R}$$

Fig. 3 Route 2 without deformation gradient $\boldsymbol{F}$

From the definition of the displacement vector variation, we have $\Delta \boldsymbol{u} = \oint_{\partial\Psi} \mathrm{d}\boldsymbol{u}$, and according to the Stokes integration theorem of exterior calculus, we have

$$\Delta \boldsymbol{u} = \oint_{\partial\Psi} \mathrm{d}\boldsymbol{u} = \iint_{\Psi} \mathrm{d}(\mathrm{d}\boldsymbol{u}) = \iint_{\Psi} \mathrm{d}^2 \boldsymbol{u}. \tag{42}$$

Starting with the basis $\boldsymbol{G}_i$ and an arbitrary displacement vector $\boldsymbol{u} = u^i \boldsymbol{G}_i$, operate on it with d twice, keeping in mind that its action on functions and differential forms is exactly the same as the exterior derivatives, thus $\mathrm{d}\boldsymbol{u} = \mathrm{d}u^i \boldsymbol{G}_i + u^i \mathrm{d}\boldsymbol{G}_i$ and according to the Poincaré lemma $\mathrm{d}^2\omega = 0$, we have

$$\mathrm{d}^2\boldsymbol{u} = \underbrace{\mathrm{d}^2 u^i}_{=0} \boldsymbol{G}_i + \underbrace{(-1)^1 \mathrm{d}u^i \wedge \mathrm{d}\boldsymbol{G}_i + \mathrm{d}u^i \wedge \mathrm{d}\boldsymbol{G}_i}_{=0} + u^i \mathrm{d}^2 \boldsymbol{G}_i = u^i \mathrm{d}^2 \boldsymbol{G}_i. \tag{43}$$

This equation has a remarkable property. It leaves the components of $\boldsymbol{u}$ undifferentiated. It appears that $\mathrm{d}^2\boldsymbol{u}$ depends not only on external objects (vector), but also on the intrinsic property of the manifold[41,45,49,51,53].

Equation (42) becomes

$$\Delta \boldsymbol{u} = \iint_{\Psi} u^i \mathrm{d}^2 \boldsymbol{G}_i. \tag{44}$$

The problem of $\Delta \boldsymbol{u}$ transforms to the calculation of $\mathrm{d}^2 \boldsymbol{G}_i$. Let us introduce connection coefficients Γ^j_{ik}, and expand the vector-valued 1-form $\mathrm{d}\boldsymbol{G}_i$ as

$$\mathrm{d}\boldsymbol{G}_i = \boldsymbol{G}_j \otimes \boldsymbol{\omega}^j_i, \tag{45}$$

where $\boldsymbol{\omega}^j_i \equiv \Gamma^j_{ik} \mathrm{d}X^k$. It is worth noting that the tensor product can be written as $\boldsymbol{G}_j \otimes \boldsymbol{\omega}^j_i = \boldsymbol{G}_j \boldsymbol{\omega}^j_i$, for convenience we have omitted the tensor product $\otimes$ in the previous formulations, and we will carry on the same policy.

Differentiating Eq. (45) once more, we obtain the vector-valued 2-form,

$$\begin{aligned} \mathrm{d}^2 \boldsymbol{G}_i &= \mathrm{d}(\boldsymbol{G}_j \boldsymbol{\omega}^j_i) \\ &= \mathrm{d}\boldsymbol{G}_j \wedge \boldsymbol{\omega}^j_i + \boldsymbol{G}_j \mathrm{d}\boldsymbol{\omega}^j_i \\ &= (\boldsymbol{G}_k \boldsymbol{\omega}^k_j) \wedge \boldsymbol{\omega}^j_i + \boldsymbol{G}_j \mathrm{d}\boldsymbol{\omega}^j_i \\ &= \boldsymbol{G}_k (\boldsymbol{\omega}^k_j \wedge \boldsymbol{\omega}^j_i) + \boldsymbol{G}_j \mathrm{d}\boldsymbol{\omega}^j_i \\ &= \boldsymbol{G}_j (\mathrm{d}\boldsymbol{\omega}^j_i + \boldsymbol{\omega}^j_k \wedge \boldsymbol{\omega}^k_i). \end{aligned} \tag{46}$$

The expression in parentheses is a 2-form, called the curvature 2-form,

$$\boldsymbol{\theta}_i^j \equiv \mathrm{d}\boldsymbol{\omega}_i^j + \boldsymbol{\omega}_k^j \wedge \boldsymbol{\omega}_i^k, \quad \text{or} \quad \boldsymbol{\theta}_{ij} = \mathrm{d}\boldsymbol{\omega}_{ij} + \boldsymbol{\omega}_{ik} \wedge \boldsymbol{\omega}_j^k. \tag{47}$$

With this notation, Eq. (46) becomes

$$\mathrm{d}^2\boldsymbol{G}_i = \boldsymbol{G}_j\boldsymbol{\theta}_i^j, \tag{48}$$

and Eq. (43) becomes

$$\mathrm{d}^2\boldsymbol{u} = u^i\mathrm{d}^2\boldsymbol{G}_i = u^i\boldsymbol{G}_j\boldsymbol{\theta}_i^j = \boldsymbol{G}_j\boldsymbol{\theta}_i^j u^i. \tag{49}$$

In Eq. (46), we have

$$\begin{aligned}\mathrm{d}\boldsymbol{\omega}_i^j &= \mathrm{d}(\Gamma_{im}^j\mathrm{d}X^m) = \mathrm{d}\Gamma_{im}^j \wedge \mathrm{d}X^m + \Gamma_{im}^j\underbrace{\mathrm{d}(\mathrm{d}X^m)}_{=0}\\ &= \Gamma_{im,l}^j\mathrm{d}X^l \wedge \mathrm{d}X^m,\end{aligned} \tag{50}$$

$$\boldsymbol{\omega}_k^j \wedge \boldsymbol{\omega}_i^k = \Gamma_{kl}^j\Gamma_{im}^k\mathrm{d}X^l \wedge \mathrm{d}X^m. \tag{51}$$

Since $\mathrm{d}X^l \wedge \mathrm{d}X^m = -\mathrm{d}X^m \wedge \mathrm{d}X^l$, the curvature 2-form $\boldsymbol{\theta}_i^j$ can be written as

$$\boldsymbol{\theta}_j^i = \frac{1}{2}R_{jkl}^i\mathrm{d}X^k \wedge \mathrm{d}X^l, \quad \text{or} \quad \boldsymbol{\theta}_{ij} = \frac{1}{2}R_{ijkl}\mathrm{d}X^k \wedge \mathrm{d}X^l, \tag{52}$$

which defines the components R_{ijkl} of the Riemann curvature tensor as in Eq. (13). Using the current index, it is defined as

$$R_{jkl}^i = \frac{\partial}{\partial X^k}\Gamma_{jl}^i - \frac{\partial}{\partial X^l}\Gamma_{jk}^i + \Gamma_{mk}^i\Gamma_{jl}^m - \Gamma_{ml}^i\Gamma_{jk}^m. \tag{53}$$

The displacement vector variation (42) can be expressed as

$$\begin{aligned}\Delta\boldsymbol{u} &= \iint_\Psi \mathrm{d}^2\boldsymbol{u} = \iint_\Psi \boldsymbol{G}_j\boldsymbol{\theta}_i^j u^i\\ &= \iint_\Psi u^i\boldsymbol{G}_j\frac{1}{2}R_{ikl}^j\mathrm{d}X^k \wedge \mathrm{d}X^l\\ &= \frac{1}{2}\iint_\Psi u^iR_{ikl}^j\boldsymbol{G}_j\mathrm{d}X^k \wedge \mathrm{d}X^l\\ &= \frac{1}{2}\iint_\Psi \boldsymbol{u}\cdot\boldsymbol{G}^iR_{ikl}^j\boldsymbol{G}_j\mathrm{d}X^k \wedge \mathrm{d}X^l\\ &= \frac{1}{2}\iint_\Psi \boldsymbol{u}\cdot R_{ikl}^j\boldsymbol{G}^i\boldsymbol{G}_j\mathrm{d}X^k \wedge \mathrm{d}X^l\\ &= \frac{1}{2}\iint_\Psi \boldsymbol{u}\cdot R_{jikl}\boldsymbol{G}^i\boldsymbol{G}^j\mathrm{d}X^k \wedge \mathrm{d}X^l.\end{aligned} \tag{54}$$

Since $\mathrm{d}X^k = \boldsymbol{G}^k \cdot \mathrm{d}\boldsymbol{X}$, the above equation becomes

$$\begin{aligned}\Delta\boldsymbol{u} &= \frac{1}{2}\iint_\Psi \boldsymbol{u}\cdot R_{jikl}\boldsymbol{G}^i\boldsymbol{G}^j\mathrm{d}X^k \wedge \mathrm{d}X^l\\ &= \frac{1}{2}\iint_\Psi \boldsymbol{u}\cdot R_{jikl}\boldsymbol{G}^i\boldsymbol{G}^j(\boldsymbol{G}^k\cdot\mathrm{d}\boldsymbol{X}) \wedge (\boldsymbol{G}^l\cdot\mathrm{d}\boldsymbol{X})\\ &= \frac{1}{2}\iint_\Psi \boldsymbol{u}\cdot R_{jikl}\boldsymbol{G}^i\boldsymbol{G}^j(\boldsymbol{G}^k \wedge \boldsymbol{G}^l)\cdot(\mathrm{d}\boldsymbol{X} \wedge \mathrm{d}\boldsymbol{X})\\ &= \frac{1}{2}\iint_\Psi \boldsymbol{u}\cdot R_{jikl}\boldsymbol{G}^i\boldsymbol{G}^j\boldsymbol{G}^k\boldsymbol{G}^l : \boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{A}.\end{aligned} \tag{55}$$

Since $R_{jikl}=-R_{ijkl}$, the above equation becomes

$$\Delta\boldsymbol{u}=-\frac{1}{2}\iint_{\Psi}\boldsymbol{u}\cdot R_{ijkl}\boldsymbol{G}^i\boldsymbol{G}^j\boldsymbol{G}^k\boldsymbol{G}^l:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{A}. \tag{56}$$

Introducing the total expression of the Riemann tensor $\boldsymbol{R}=R_{ijkl}\boldsymbol{G}^i\boldsymbol{G}^j\boldsymbol{G}^k\boldsymbol{G}^l$, we have a theorem about the displacement vector variation as follows.

Theorem 4 *Let $\boldsymbol{u}$ be the displacement vector, $\partial\Psi$ be a closed contour/curve of a surface Ψ. Thus, the total change or variation $\Delta\boldsymbol{u}$ on going around $\partial\Psi$ is given by*

$$\Delta\boldsymbol{u}=-\frac{1}{2}\iint_{\Psi}\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{A}. \tag{57}$$

It is worth emphasizing again that the formulation of Eq. (57) on Route 1 does not use the deformation gradient $\boldsymbol{F}$, which is a pure differential geometric approach. Fortunately, the result obtained in Eq. (57) is exactly the same as Eqs. (12) and (33) by using the deformation gradient $\boldsymbol{F}$ on Route 1.

Finally, we answer the mentioned question in the beginning of this section. Route 1 and Route 2 will lead to the same result.

6 Conclusions

This article revisits the compatibility conditions of the deformation field in continuum mechanics. The explicit total tensor expression between the displacement vector variation $\Delta\boldsymbol{u}$ and the Riemannian curvature tensors $\boldsymbol{R}$ is obtained (see Fig. 4). The study shows that $\Delta\boldsymbol{u}$ is linearly proportional to the Riemannian curvature tensor $\boldsymbol{R}$ and displacement vector $\boldsymbol{u}$.

$$\begin{tikzcd}
\Delta\boldsymbol{u} \arrow[r, "\text{Route 2}"] \arrow[d, "\text{Route 1}"'] & \oint\mathrm{d}\boldsymbol{u} \arrow[d] \arrow[dl, dotted, leftrightarrow] \\
\oint(\boldsymbol{F}-\boldsymbol{I})\cdot\mathrm{d}\boldsymbol{X} \arrow[r] & -\frac{1}{2}\iint_{\Psi}\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{A}
\end{tikzcd}$$

Fig. 4 $\Delta\boldsymbol{u}$ linearly proportional to $\boldsymbol{R}$

The explicit expression reconfirms that the compatibility condition is equivalent to the vanishing of the Riemann tensor. The Burgers vector has been given as $\mathrm{d}\boldsymbol{b}=-\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{A}$, and the dislocation density tensor has been expressed as $\boldsymbol{T}=-\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{R}:\boldsymbol{\varepsilon}$. Finally, the mathematical formula of a change on a vector which is parallel displaced around a closed curve has been reformulated and verified in a most abstract approach in exterior differential forms.

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Appendix A

Notation and preliminaries on finite deformation field

To distinguish between the undeformed and deformed states, the quantities with the undeformed body (state) $\mathfrak{B}$ will be denoted by the upper case (majuscules), and those associated with the deformed body (state) $\mathfrak{b}$ by the lower case (minuscules). When these quantities are referred to Lagrange coordinates X^K, their indices will be the upper case (majuscules), and when they are referred to Euler x^k, their indices will be the lower case (minuscules). For example, a displacement vector $\boldsymbol{u}$ referred to X^K will have components u_K, and referred to x^k will have the components u_k. If we denote $\boldsymbol{G}_K$ and $\boldsymbol{g}_k$ as the covariant base vector in the undeformed body and the deformed body, respectively, then we can write the displacement vector in a total form, $\boldsymbol{u} = u^K\boldsymbol{G}_K = u^k\boldsymbol{g}_K$.

In the undeformed state, let $\boldsymbol{X} = \boldsymbol{X}(X^A)$ be the position vector of a particle and X^A be Lagrange coordinates of the particle, then, its differential is $\mathrm{d}\boldsymbol{X} = \frac{\partial \boldsymbol{X}}{\partial X^A}\mathrm{d}X^A = \boldsymbol{X}\boldsymbol{\nabla}_{X^A}\mathrm{d}X^A = \boldsymbol{X}\boldsymbol{\nabla}_A\mathrm{d}X^A =$

$\boldsymbol{G}_A \mathrm{d}X^A$, where $\boldsymbol{\nabla}_A = \boldsymbol{G}^A \frac{\partial}{\partial X^A}$ and $\boldsymbol{G}_A$ are the covariant derivative and tangent vector in the underformed state, respectively. After the time t, the position moves to the deformed state, its position is $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$, and its differential is $\mathrm{d}\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial x^i}\mathrm{d}x^i = \boldsymbol{x}\boldsymbol{\nabla}_{x^i}\mathrm{d}x^i = \boldsymbol{x}\boldsymbol{\nabla}_i \mathrm{d}x^i = \boldsymbol{g}_i \mathrm{d}x^i$, where $\boldsymbol{\nabla}_i = \boldsymbol{g}^i \frac{\partial}{\partial x^i}$ and $\boldsymbol{g}_i$ are the gradient operator and tangent vector in the deformed state, respectively.

Let $\mathrm{d}\boldsymbol{X}$ be the line element between two particles X^A and $X^A + \mathrm{d}X^A$, after the deformation, the line element becomes $\mathrm{d}\boldsymbol{x}$ between the corresponding particles $\mathrm{d}x^i$ and $x^i + \mathrm{d}x^i$, then $\mathrm{d}\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}, t) - \boldsymbol{x}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{X} = \boldsymbol{x}\boldsymbol{\nabla}_A \cdot \mathrm{d}\boldsymbol{X} = \frac{\partial \boldsymbol{x}}{\partial X^A}\frac{\partial X^A}{\partial \boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{X} = \frac{\partial \boldsymbol{x}}{\partial X^A}\boldsymbol{G}^A \cdot \mathrm{d}\boldsymbol{X} = \frac{\partial \boldsymbol{x}}{\partial x^i}\frac{\partial x^i}{\partial X^A}\boldsymbol{G}^A \cdot \mathrm{d}\boldsymbol{X} = \boldsymbol{g}_i \frac{\partial x^i}{\partial X^A}\boldsymbol{G}^A \cdot \mathrm{d}\boldsymbol{X} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{X} \cdot \boldsymbol{F}^{\mathrm{T}}$, where the deformation gradient tensor $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{x}\boldsymbol{\nabla}_A = \frac{\partial \boldsymbol{x}}{\partial X^A}\boldsymbol{G}^A = \boldsymbol{g}_i F^i_A \boldsymbol{G}^A = F^i_A \boldsymbol{g}_i \boldsymbol{G}^A$, its components $F^i_A = \frac{\partial x^i}{\partial X^A} = x^i_{;A} = \nabla_A x^i$, $\boldsymbol{F}^{\mathrm{T}} = \frac{\partial \boldsymbol{x}}{\partial X^A}\boldsymbol{G}^A = F^i_A \boldsymbol{G}^A \boldsymbol{g}_i$, and $\boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial x^i}\boldsymbol{g}^i = F_i^A \boldsymbol{G}_A \boldsymbol{g}^i$.

Let $\boldsymbol{u}$ be the displacement vector, then, $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{X}$, the deformation gradient tensor $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{X}} = \boldsymbol{I} + \boldsymbol{g}_i \boldsymbol{G}^A \nabla_A u^i$, the transpose $\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{I} + \boldsymbol{G}^A \boldsymbol{g}_i \nabla_A u^i$ and the inverse $\boldsymbol{F}^{-1} = \boldsymbol{I} - \boldsymbol{u}\nabla_{\boldsymbol{x}} = \boldsymbol{I} - \boldsymbol{g}^j \boldsymbol{g}^i \nabla_i u^j$.

The materials time derivative of $\mathrm{d}\boldsymbol{x}$ leads to $\mathrm{d}\dot{\boldsymbol{x}} = \dot{\boldsymbol{F}} \cdot \mathrm{d}\boldsymbol{X} = \boldsymbol{l}\boldsymbol{F}\mathrm{d}\boldsymbol{x}$, in which the velocity gradient tensor is defined as $\boldsymbol{l} = \dot{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{x}} = \frac{\partial \dot{u}}{\partial x} = \dot{u}^i_{;j}\boldsymbol{g}_i \boldsymbol{g}^j = \nabla_j u^i \boldsymbol{g}_i \boldsymbol{g}^j = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, which can be decomposed as the sum of $\boldsymbol{d}$ and $\boldsymbol{w}$, $\boldsymbol{l} = \boldsymbol{d} + \boldsymbol{w}$, where the rate of deformation tensor $\boldsymbol{d} = \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^{\mathrm{T}})$ and the spin tensor $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{l} - \boldsymbol{l}^{\mathrm{T}})$.

If we denote $\mathrm{d}\boldsymbol{A}$ as the area element in the undeformed state, then the area in the deformed one $\mathrm{d}\boldsymbol{a} = J\boldsymbol{F}^{-\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}$ and $\mathrm{d}\boldsymbol{A} = J^{-1}\boldsymbol{F}^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{a}$, where the Jacobean $J = \det(\boldsymbol{F})$.

The metric tensor in the undeformed body $G_{AB} = \boldsymbol{G}_A \boldsymbol{G}_B$ and in the deformed body $g_{ij} = \boldsymbol{g}_i \boldsymbol{g}_j$.

The tangent vectors between the undeformed and deformed state can be easily transferred as $\boldsymbol{g}_i = \delta_{iA}\boldsymbol{G}^A$, $\boldsymbol{g}^i = \delta^{iA}\boldsymbol{G}_A$, $\boldsymbol{G}_A = \delta_{Ai}\boldsymbol{g}^i$, $\boldsymbol{G}^A = \delta^{Ai}\boldsymbol{g}_i$, where the shifters $\delta_{iA} = \boldsymbol{g}_i \cdot \boldsymbol{G}_A = \delta_{Ai}$, $\delta^{iA} = \boldsymbol{g}^i \cdot \boldsymbol{G}^A = \delta^{Ai}$.

Appendix B

Curl of infinitesimal rotation tensor

The displacement gradient $\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{X}}$ can be expressed as the sum of a symmetric tensor and an antisymmetric tensor $\boldsymbol{u}\boldsymbol{\nabla} = \boldsymbol{\epsilon} + \boldsymbol{\Omega}$, where the symmetric part $\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{u}\boldsymbol{\nabla} + (\boldsymbol{u}\boldsymbol{\nabla})^{\mathrm{T}})$ is the infinitesimal strain tensor, and the antisymmetric part $\boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{u}\boldsymbol{\nabla} - (\boldsymbol{u}\boldsymbol{\nabla})^{\mathrm{T}})$, which is known as the infinitesimal rotation tensor with the property $\boldsymbol{\Omega}^{\mathrm{T}} = -\boldsymbol{\Omega}$. Thus, there are three independent components of $\boldsymbol{\Omega}$,

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & -\Omega_{12} & -\Omega_{13} \\ \Omega_{12} & 0 & -\Omega_{23} \\ \Omega_{13} & \Omega_{23} & 0 \end{pmatrix}. \tag{B1}$$

Here, there is no restriction that is placed on the magnitude of $\boldsymbol{u}\boldsymbol{\nabla}$. Generally speaking, $\boldsymbol{\epsilon}$ and $\boldsymbol{\Omega}$ do not have the meaning of the infinitesimal strain and the infinitesimal rotation tensors, unless the deformation $\boldsymbol{u}$ is infinitesimal.

Since $\boldsymbol{\Omega}$ only has three independent components, the three components can be used to define the components of a vector $\boldsymbol{\omega}$,

$$\boldsymbol{\Omega} = -\varepsilon \cdot \boldsymbol{\omega}\boldsymbol{\omega} = -\frac{1}{2}\varepsilon : \boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{u} \times \boldsymbol{\nabla}_{\boldsymbol{X}} = \frac{1}{2}\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{X}} : \varepsilon. \tag{B2}$$

Thus,

$$\varepsilon \cdot (\boldsymbol{\omega} \times \boldsymbol{\nabla}_{\boldsymbol{X}}) = -\varepsilon \cdot (\boldsymbol{\omega}\boldsymbol{\nabla}_{\boldsymbol{X}}) : \varepsilon = (\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u}) : \varepsilon = \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R} : \varepsilon, \tag{B3}$$

then,

$$\varepsilon \cdot (\boldsymbol{\omega}\boldsymbol{\nabla}_{\boldsymbol{X}}) = -\frac{1}{2}\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u} = -\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{R}, \tag{B4}$$

or

$$\boldsymbol{\omega}\boldsymbol{\nabla}_{\boldsymbol{X}} = -\frac{1}{4}\varepsilon : (\boldsymbol{R}(\boldsymbol{G}_A, \boldsymbol{G}_B)\boldsymbol{u}) = -\frac{1}{4}\varepsilon : (\boldsymbol{u} \cdot \boldsymbol{R}). \tag{B5}$$

This means that the rotation gradient $\omega\boldsymbol{\nabla}_{\boldsymbol{X}}$ will generate the curvature $\boldsymbol{R}$ and vice versa.

Appendix C

Formulations in deformed state $\mathfrak{b}$

C1 Displacement vector variation In the deformed state $\mathfrak{b}$, there is a surface ψ with a closed boundary $\partial\psi$, and the displacement vector variation along the closed loop $\partial\psi$ can be written as

$$\begin{aligned}
\Delta\boldsymbol{u} &= \oint_{\partial\psi} \mathrm{d}\boldsymbol{u} = \oint_{\partial\psi} \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}} \cdot \mathrm{d}\boldsymbol{x} = \oint_{\partial\psi} (\mathbf{1} - \boldsymbol{F}^{-1}) \cdot \mathrm{d}\boldsymbol{x} \\
&= \oint_{\partial\psi} (\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{x}}) \cdot \mathrm{d}\boldsymbol{x} = -\iint_{\psi} (\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{x}}) \times \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \mathrm{d}\boldsymbol{a} \\
&= -\iint_{\psi} (\boldsymbol{u}\nabla_i \boldsymbol{g}^i) \times \nabla_j \boldsymbol{g}^j \cdot \mathrm{d}\boldsymbol{a} \\
&= -\iint_{\psi} (\boldsymbol{u}\nabla_i \nabla_j) \boldsymbol{g}^i \times \boldsymbol{g}^j \cdot \mathrm{d}\boldsymbol{a} \\
&= -\frac{1}{2} \iint_{\psi} (\boldsymbol{u}\nabla_i\nabla_j - \boldsymbol{u}\nabla_j\nabla_i) \boldsymbol{g}^i \boldsymbol{g}^j : \boldsymbol{\varepsilon} \cdot \mathrm{d}\boldsymbol{a} \\
&= -\frac{1}{2} \iint_{\psi} \boldsymbol{u}\boldsymbol{r}(\boldsymbol{g}_i, \boldsymbol{g}_j) : \boldsymbol{\varepsilon} \cdot \mathrm{d}\boldsymbol{a},
\end{aligned} \tag{C1}$$

where the permutation tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{ijk} \boldsymbol{g}^i \boldsymbol{g}^j \boldsymbol{g}^k$ and

$$\begin{aligned}
\boldsymbol{u}\boldsymbol{r}(\boldsymbol{g}_i, \boldsymbol{g}_j) &= (\boldsymbol{u}\nabla_i\nabla_j - \boldsymbol{u}\nabla_j\nabla_i) \boldsymbol{g}^i \boldsymbol{g}^j \\
&= \boldsymbol{g}^k (u_k\nabla_i\nabla_j - u_k\nabla_j\nabla_i) \boldsymbol{g}^i \boldsymbol{g}^j \\
&= \boldsymbol{g}^k u_l r^l_{.kij} \boldsymbol{g}^i \boldsymbol{g}^j = \boldsymbol{u} \cdot \boldsymbol{g}_l \boldsymbol{g}^k r^l_{.kij} \boldsymbol{g}^i \boldsymbol{g}^j \\
&= \boldsymbol{u} \cdot r^l_{.kij} \boldsymbol{g}_l \boldsymbol{g}^k \boldsymbol{g}^i \boldsymbol{g}^j \\
&= \boldsymbol{u} \cdot \boldsymbol{r},
\end{aligned} \tag{C2}$$

where the Riemann-Christoffel curvature tensor $\boldsymbol{r} = r^l_{.kij} \boldsymbol{g}_l \boldsymbol{g}^k \boldsymbol{g}^i \boldsymbol{g}^j$, which is defined in the deformed state with its components $r^l_{.kij} := \partial_i \gamma^l_{jk} - \partial_j \gamma^l_{ik} + \gamma^l_{im}\gamma^m_{jk} - \gamma^l_{jm}\gamma^m_{ik}$. The symbols γ^l_{jk} are called the coefficients of the affine connections, or the Christoffel symbols, with respect to the frame $\boldsymbol{g}_j$, that is, $\boldsymbol{\nabla}_{\boldsymbol{g}_j} \boldsymbol{g}_k = \boldsymbol{g}_l \gamma^l_{jk}$.

Thus, the displacement vector variation in Eq. (C1) will be

$$\Delta\boldsymbol{u} = -\frac{1}{2} \iint_{\Psi} \boldsymbol{u} \cdot \boldsymbol{r} : \boldsymbol{\varepsilon} \cdot \mathrm{d}\boldsymbol{a}. \tag{C3}$$

Therefore, the curl of the displacement gradient in the deformed state is

$$(\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{x}}) \times \boldsymbol{\nabla}_{\boldsymbol{x}} = \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{r} : \boldsymbol{\varepsilon}, \tag{C4}$$

or

$$(\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{x}})\boldsymbol{\nabla}_{\boldsymbol{x}} = \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{r}. \tag{C5}$$

The displacement density flux tensor $\boldsymbol{t}$ in the deformed state is

$$\boldsymbol{t} = \frac{\mathrm{d}\Delta\boldsymbol{u}}{\mathrm{d}\boldsymbol{a}} = -\frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{r} : \boldsymbol{\varepsilon}. \tag{C6}$$

It is clear to see that the compatibility condition in the deformed state $\mathfrak{b}$ is $\Delta\boldsymbol{u} = \mathbf{0}$, which is equivalent to the vanishing of the Riemann curvature tensor, $\boldsymbol{r} = r^l_{.kij} \boldsymbol{g}_l \boldsymbol{g}^k \boldsymbol{g}^i \boldsymbol{g}^j = \mathbf{0}$, in the component form, $r^l_{.kij} = 0$.

C2 Rate of deformation gradient tensor   Making a material derivative of Eq. (C1) with respect to the time,

$$\begin{aligned}\Delta\dot{\boldsymbol{u}} &= \oint_{\partial\psi}\mathrm{d}\dot{\boldsymbol{u}} = \oint_{\partial\psi}\frac{\partial\dot{\boldsymbol{u}}}{\partial\boldsymbol{x}}\cdot\mathrm{d}\boldsymbol{x} = \oint_{\partial\psi}(\nabla_i\dot{\boldsymbol{u}})\boldsymbol{g}^i\cdot\mathrm{d}\boldsymbol{x} = \oint_{\partial\psi}\boldsymbol{l}\cdot\mathrm{d}\boldsymbol{x}\\ &= \oint_{\partial\psi}(\boldsymbol{d}+\boldsymbol{w})\cdot\mathrm{d}\boldsymbol{x} = -\iint_{\psi}(\boldsymbol{d}+\boldsymbol{w})\times\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\mathrm{d}\boldsymbol{a} = -\iint_{\psi}\boldsymbol{w}\times\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\mathrm{d}\boldsymbol{a}\\ &= -\iint_{\psi}(\nabla_i\nabla_j\dot{\boldsymbol{u}})\boldsymbol{g}^i\times\boldsymbol{g}^j\cdot\mathrm{d}\boldsymbol{x} = -\frac{1}{2}\iint_{\psi}(\nabla_i\nabla_j\dot{\boldsymbol{u}}-\nabla_j\nabla_i\dot{\boldsymbol{u}})\boldsymbol{g}^i\times\boldsymbol{g}^j\cdot\mathrm{d}\boldsymbol{x}\\ &= -\frac{1}{2}\iint_{\psi}\boldsymbol{g}^k(\nabla_i\nabla_j\dot{u}_k-\nabla_j\nabla_i\dot{u}_k)\boldsymbol{g}^i\boldsymbol{g}^j:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{a}\\ &= -\frac{1}{2}\iint_{\psi}\dot{\boldsymbol{u}}\cdot\boldsymbol{r}:\boldsymbol{\varepsilon}\cdot\mathrm{d}\boldsymbol{a},\end{aligned}\tag{C7}$$

then, the curl of the rate of deformation tensor is

$$\boldsymbol{l}\times\boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{w}\times\boldsymbol{\nabla}_{\boldsymbol{x}} = \frac{1}{2}\dot{\boldsymbol{u}}\cdot\boldsymbol{r}:\boldsymbol{\varepsilon}.\tag{C8}$$

From $\boldsymbol{w}\times\boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{w}\boldsymbol{\nabla}_{\boldsymbol{x}}:\boldsymbol{\varepsilon}$, thus,

$$\boldsymbol{l}\boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{w}\boldsymbol{\nabla}_{\boldsymbol{x}} = \frac{1}{2}\dot{\boldsymbol{u}}\cdot\boldsymbol{r}.\tag{C9}$$

C3 In exterior differential forms   Regarding the exterior differential forms, we have the displacement vector variation,

$$\Delta\boldsymbol{u} = \oint_{\partial\psi}\mathrm{d}\boldsymbol{u} = \iint_{\Psi}\mathrm{d}(\mathrm{d}\boldsymbol{u}) = \iint_{\Psi}\mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{y}),\tag{C10}$$

in which

$$\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{y} = (\boldsymbol{I}-\boldsymbol{F}^{-1})\cdot\mathrm{d}\boldsymbol{y}\tag{C11}$$

is a vector-valued 1-form, and differentiating the above equation once more, we obtain the vector-valued 2-form,

$$\begin{aligned}\mathrm{d}\boldsymbol{F}^{-1} &= \mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{y})\\ &= \mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u})\wedge\mathrm{d}\boldsymbol{y}+(-1)^0\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}\mathrm{d}^2\boldsymbol{y}\\ &= \boldsymbol{\nabla}_{\boldsymbol{y}}(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y}\\ &= \boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y},\end{aligned}\tag{C12}$$

where Ref. [48] is used for $\mathrm{d}^2\boldsymbol{y}=\boldsymbol{0}$.

Due to the antisymmetric nature of exterior algebra, $\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y} = -\mathrm{d}\boldsymbol{y}\wedge\mathrm{d}\boldsymbol{x}$,

$$\mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{y}) = -\frac{1}{2}(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y}.\tag{C13}$$

According to the definition of the Riemann tensor,

$$\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u} = \boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}-\boldsymbol{\nabla}_{[\boldsymbol{x},\boldsymbol{y}]}\boldsymbol{u},\tag{C14}$$

and in the coordinate frame $\boldsymbol{\nabla}_{[\boldsymbol{x},\boldsymbol{y}]}\boldsymbol{u}=\boldsymbol{0}$, we have

$$\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u} = \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}.\tag{C15}$$

If we expand the vector fields in terms of the coordinate basis ∂_i, the Riemann tensor $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u} = (r^i_{jkl}x^ky^lu^j)\partial_i$ and its components $r^i_{jkl} := \partial_k\gamma^i_{lj}-\partial_l\gamma^i_{kj}+\gamma^I_{km}\gamma^m_{lj}-\gamma^I_{LM}\gamma^m_{kj}$. The symbols γ^i_{jk} are called the coefficients of the affine connections, or the Christoffel symbols, with respect to the frame $\boldsymbol{g}_j$, that is, $\boldsymbol{\nabla}_{\boldsymbol{g}_j}\boldsymbol{g}_k = \boldsymbol{g}_i\gamma^i_{jk}$.

Therefore, we have

$$\begin{aligned}\mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{y}) &= -\frac{1}{2}(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}-\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y}\\ &= -\frac{1}{2}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y}.\end{aligned}\tag{C16}$$

Finally, we have the displacement change in a differential form,

$$\begin{aligned}\Delta \boldsymbol{u} &= \oint_{\partial\psi} \mathrm{d}\boldsymbol{u} = \iint_{\Psi} \mathrm{d}(\mathrm{d}\boldsymbol{u}) = \iint_{\Psi} \mathrm{d}(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{y}) \\ &= -\frac{1}{2}\iint_{\Psi} (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{u} - \boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u})\cdot \mathrm{d}\boldsymbol{x}\wedge \mathrm{d}\boldsymbol{y} \\ &= -\frac{1}{2}\iint_{\Psi} \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{x}\wedge \mathrm{d}\boldsymbol{y}. \end{aligned} \tag{C17}$$

Note that $\mathrm{d}\boldsymbol{a} = \mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y}$, hence, the displacement density flux $\boldsymbol{t}$ and the incompatibility operator $\mathrm{inc}(\boldsymbol{F})$

$$\boldsymbol{t} = \frac{\mathrm{d}\Delta\boldsymbol{u}}{\mathrm{d}\boldsymbol{a}} = -\frac{1}{2}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}, \tag{C18}$$

from $\mathrm{d}\boldsymbol{F}^{-1}\wedge \mathrm{d}\boldsymbol{y} = -(1/2)\boldsymbol{r}\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\boldsymbol{y}$, then,

$$\mathrm{d}\boldsymbol{F}^{-1} = -\frac{1}{2}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{x}. \tag{C19}$$

Making a materials time derivative for the above equation, we have the rate of deformation gradient,

$$\mathrm{d}\boldsymbol{l} = -\frac{1}{2}\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\dot{\boldsymbol{u}}\cdot \mathrm{d}\boldsymbol{x}. \tag{C20}$$

Using the language of differential forms, the compatibility conditions in the deformed state $\mathfrak{b}$ can be stated as follows:

(i) Compatibility conditions of simply-connected bodies

For simply-connected bodies, the displacement change integral along the closed loop must be zero, $\mathrm{inc}(\boldsymbol{F}) = \boldsymbol{0}$, therefore, the Riemann tensor $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})$ must vanish because of the arbitrary nature of the displacement $\boldsymbol{u}$.

(ii) Compatibility conditions of non-simply-connected bodies

For non-simply-connected bodies, the vanishing of Riemann tensor is not enough and needs extra conditions. According to the de Rham theorem[47,51–52], these extra conditions are $\oint_{\partial\psi_i}\mathrm{d}\boldsymbol{u} = \oint_{\partial\psi_i}(\boldsymbol{1} - \boldsymbol{F}^{-1})\cdot \mathrm{d}\boldsymbol{x} = \boldsymbol{0}$, where $\partial\psi_i$ are the closed loops, including the holes and/or defects.