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ABSTRACT

To avoid the infinitesimal rotation nature of the Cauchy-Stokes decomposition of velocity gradient, the letter proposes an new additive decomposition in which one part is a SO(3) rotation tensor $\boldsymbol{Q} = \exp \boldsymbol{W}$.

The vortex identification is a quite important tool for turbulence study.[1–7] Recently, to have a better vortex identification, a vector named rortex vector was proposed.[8–13] The basic argument is that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ would not able to represent vortex (rotation) in fluid and should be further decomposed into two parts, namely,

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \boldsymbol{R} + \boldsymbol{S}. \tag{1}$$

One is the rotational part $\boldsymbol{R}$, which is contributed to fluid rotation, and the other is nonrotational part $\boldsymbol{S}$, contributed to shear. This rotational part $\boldsymbol{R}$ is defined as a rortex vector, and different kinds of rortex vectors in Cartesian coordinates were proposed.[8–13]

Numerical simulations[8–13] indicate that the rortex vector might be a promising and/or better quantity for vortex identification. If it is true, the question has become to find a general formulation of $\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \boldsymbol{R} + \boldsymbol{S}$ which should be valid in any coordinate system.

Because the vorticity, $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$, can be expressed in terms of velocity gradient $\nabla \boldsymbol{v}$, namely,

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \varepsilon : \nabla \boldsymbol{v}, \tag{2}$$

where the permutation tensor (symbol) $\varepsilon = \varepsilon_{ijk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k$, and $\boldsymbol{e}_k$ the base vector. This can be verified easily as follows: $\varepsilon : \nabla \boldsymbol{v} = (\varepsilon_{ijk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k):\boldsymbol{e}_p\nabla_p(v_q\boldsymbol{e}_q) = \varepsilon_{ijk}(\nabla_p v_q)\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k:\boldsymbol{e}_p\boldsymbol{e}_q = \varepsilon_{ijk}(\nabla_p v_q)\boldsymbol{e}_i(\boldsymbol{e}_j \cdot \boldsymbol{e}_p)(\boldsymbol{e}_k \cdot \boldsymbol{e}_q) = \varepsilon_{ijk}(\nabla_p v_q)\boldsymbol{e}_i\delta_{jp}\delta_{kq} = \varepsilon_{ipq}(\nabla_p v_q)\boldsymbol{e}_i = (\nabla_p v_q)\boldsymbol{e}_p \times \boldsymbol{e}_q = (\boldsymbol{e}_p\nabla_p) \times (v_q\boldsymbol{e}_q) = \nabla \times \boldsymbol{v}$.

Therefore, to make the additive decomposition $\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \boldsymbol{R} + \boldsymbol{S}$ possible, the velocity gradient $\nabla \boldsymbol{v}$ has to be decomposed additively as well since the permutation tensor ε is a constant symbolic tensor. Otherwise, the decomposition $\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \varepsilon : \nabla \boldsymbol{v} = \boldsymbol{R} + \boldsymbol{S}$ would not be compatible with the additive decomposition of the vorticity in Eq. (1). It means that the decomposition of the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ has become to a question of how to decompose the velocity gradient $\nabla \boldsymbol{v}$ additively.

Mathematically speaking, there is no unique additive decomposition of a tensor. The proper decomposition can only be defined if such decomposition works.

As the first additive decomposition, the velocity gradient $\nabla \boldsymbol{v}$ is decomposed to a symmetric part $\boldsymbol{D} = \boldsymbol{D}^T$ representing deformation rate and a skew-symmetric part $\boldsymbol{W}^T = -\boldsymbol{W}$ representing spin, namely,

$$\nabla \boldsymbol{v} = \boldsymbol{D} + \boldsymbol{W}, \tag{3}$$

which is called the Cauchy-Stokes decomposition, and $\boldsymbol{D} = \frac{1}{2}[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T]$ and $\boldsymbol{W} = \frac{1}{2}[\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T]$.

For the symmetric tensor $\boldsymbol{D}$, since $\varepsilon : \boldsymbol{D} = (D_{23} - D_{32})\boldsymbol{e}_1 + (D_{31} - D_{13})\boldsymbol{e}_2 + (D_{12} - D_{21})\boldsymbol{e}_3 = 0$, the vorticity $\boldsymbol{\omega}$ in Eq. (2) can be expressed as follows:

$$\begin{aligned}\boldsymbol{\omega} &= \varepsilon : \boldsymbol{D} + \varepsilon : \boldsymbol{W} \\ &= \varepsilon : \boldsymbol{W}.\end{aligned} \tag{4}$$

From deformation geometry analysis, it has shown that the skew tensor $\boldsymbol{W}$ can only represent the infinitesimal rotation, where is the weak point of the Cauchy-Stokes decomposition that has been criticized.[8–13]

The spin tensor $\boldsymbol{W}$ is not a proper SO(3) rotational tensor because it does not satisfies the SO(3) definition

$$\boldsymbol{W}\cdot\boldsymbol{W}^T = \boldsymbol{W}^T\cdot\boldsymbol{W} = -\boldsymbol{W}\cdot\boldsymbol{W} = -\boldsymbol{W}^2 \neq \mathbf{1}. \tag{5}$$

It reveals that the spin $\boldsymbol{W}$ is not a proper candidate for finite rotation motion formulation. The SO(3) orthogonal rotation tensor $\boldsymbol{Q}$ is defined as follows:

$$SO(3) = \{\boldsymbol{Q} : R^3 \to TR^3 | \boldsymbol{Q}\cdot\boldsymbol{Q}^T = \boldsymbol{Q}^T\cdot\boldsymbol{Q} = \mathbf{1}, \quad \det\boldsymbol{Q} = +1\}. \tag{6}$$

To avoid the infinitesimal rotation, it would be a natural to generalize the spin $\boldsymbol{W}$ to a rotation tensor $\boldsymbol{Q}$ that must be valid for any rotation instead of only for infinitesimal rotation.

In mathematics, given an antisymmetric tensor $\boldsymbol{A}$, its exponential map, i.e., $\exp : \mathfrak{so}(3) \to SO(3); \boldsymbol{A} \mapsto e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \boldsymbol{A}^k/k!$, is always in SO(3), where $\mathfrak{so}(3)$ is Lie algebra of SO(3) and consists of all skew-symmetric 3×3 tensors. The proof uses the elementary properties of the tensor exponential

$$(e^{\boldsymbol{A}})^T\cdot e^{\boldsymbol{A}} = e^{\boldsymbol{A}^T}\cdot e^{\boldsymbol{A}} = e^{-\boldsymbol{A}+\boldsymbol{A}} = e^{\boldsymbol{A}-\boldsymbol{A}} = e^{\boldsymbol{A}}\cdot(e^{\boldsymbol{A}})^T = \mathbf{1}, \tag{7}$$

and $\det(e^{\boldsymbol{A}}) = e^{tr\boldsymbol{A}} = e^0 = 1$. In general, skew-symmetric tensors over the field of real numbers form the tangent space to the real orthogonal group $O(n)$ at the identity tensor; formally, the special orthogonal Lie algebra. In this sense, then, skew-symmetric matrices can be thought of as infinitesimal rotations. Another way of saying this is that the space of skew-symmetric tensors forms the Lie algebra $\mathfrak{so}(n)$ of the Lie group $O(n)$. The Lie bracket on this space is given by the commutator $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{B}\cdot\boldsymbol{A}$. It is easy to check that the commutator of two skew-symmetric tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is again skew-symmetric, i.e., $[\boldsymbol{A}, \boldsymbol{B}]^T = (\boldsymbol{A}\cdot\boldsymbol{B})^T - (\boldsymbol{B}\cdot\boldsymbol{A})^T = (-\boldsymbol{B})\cdot(-\boldsymbol{A}) - (-\boldsymbol{A})\cdot(-\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{B} = -[\boldsymbol{A}, \boldsymbol{B}]$. The tensor exponential of a skew-symmetric tensor $\boldsymbol{A}$ is then an orthogonal tensor $\boldsymbol{Q} = e^{\boldsymbol{A}} = \sum_{k=0}^{\infty}\boldsymbol{A}^k/k!$.

Based on the above mathematical understanding, we know that the rotation tensor $\boldsymbol{Q}$ must be a tensorial function of an antisymmetric tensor. For the velocity gradient $\nabla\boldsymbol{v}$, only an antisymmetric tensor associated with the velocity gradient $\nabla\boldsymbol{v}$ is the spin tensor $\boldsymbol{W}$. Therefore, the rotation tensor $\boldsymbol{Q}$ should be an isotropic tensorial function of the spin tensor $\boldsymbol{W}$, namely, $\boldsymbol{Q} = \boldsymbol{f}(\boldsymbol{W})$. From the Cayley-Hamilton tensor representation theory,[14,15] for the 3D spin tensor $\boldsymbol{W}$, the isotropic tensor function must be in following form:

$$\boldsymbol{Q} = \boldsymbol{f}(\boldsymbol{W}) = \alpha_0\mathbf{1} + \alpha_1\boldsymbol{W} + \alpha_2\boldsymbol{W}^2, \tag{8}$$

where α_k, $k = 1, 2, 3$ can be determined later.

Mathematics suggests the rotation must be an exponential map: $e^{\beta\boldsymbol{W}}$, where β is an arbitrary constant. If we set $\beta = 1$, the simplest rotational tensor $\boldsymbol{Q}$ is hence proposed by the spin tensor $\boldsymbol{W}$ as follows:

$$\boldsymbol{Q} = e^{\boldsymbol{W}} = \exp\left\{\frac{1}{2}[\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T]\right\}. \tag{9}$$

Hence, we have an expression

$$\boldsymbol{Q} = e^{\boldsymbol{W}} = \alpha_0\mathbf{1} + \alpha_1\boldsymbol{W} + \alpha_2\boldsymbol{W}^2. \tag{10}$$

The eigenvalues of the spin tensor $\boldsymbol{W}$ are given by

$$\lambda_0 = 0,\ \lambda_2 = i\omega,\ \lambda_3 = -i\omega, \tag{11}$$

where $\omega = \sqrt{\boldsymbol{\omega}\cdot\boldsymbol{\omega}}$. Thus, substituting the eigenvalues in Eq. (10), we have

$$e^0 = \alpha_0 1 + \alpha_1(0) + \alpha_2(0)^2, \tag{12}$$

$$e^{i\omega} = \alpha_0 1 + \alpha_1(i\omega) + \alpha_2(i\omega)^2, \tag{13}$$

$$e^{-i\omega} = \alpha_0 1 + \alpha_1(-i\omega) + \alpha_2(-i\omega)^2. \tag{14}$$

By solving the equations system, we obtain the coefficients

$$\alpha_0 = 1,\ \alpha_1 = \frac{\sin\omega}{\omega},\ \alpha_2 = \frac{1-\cos\omega}{\omega^2}. \tag{15}$$

The rotation tensor in Eq. (9) can then be constructed in terms of spin tensor $\boldsymbol{W}$ as follows:

$$\begin{aligned}\boldsymbol{Q} &= \mathbf{1} + \frac{\sin\omega}{\omega}\boldsymbol{W} + \frac{1-\cos\omega}{\omega^2}\boldsymbol{W}^2\\ &= \mathbf{1} + \frac{\sin\omega}{2\omega}[\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T] + \frac{1-\cos\omega}{4\omega^2}[\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T]^2.\end{aligned} \tag{16}$$

With the rotation tensor $\boldsymbol{Q}$ defined in Eq. (16), we can propose a new additive decomposition of the velocity gradient $\nabla\boldsymbol{v}$ as follows:

$$\nabla\boldsymbol{v} = \boldsymbol{K} + \boldsymbol{Q} \tag{17}$$

and $\boldsymbol{K} = \nabla\boldsymbol{v} - \boldsymbol{Q}$, which can be expressed in terms of $\boldsymbol{D}$ and $\boldsymbol{W}$, namely,

$$\boldsymbol{K} = \boldsymbol{D} - \mathbf{1} - \frac{1-\cos\omega}{\omega^2}\boldsymbol{W}^2 + (1 - \frac{\sin\omega}{\omega})\boldsymbol{W}. \tag{18}$$

Therefore, together with Eqs. (16)–(18), the velocity gradient $\nabla\boldsymbol{v}$ has been successfully split into two parts. The additive decomposition in Eq. (17) is valid for any rotation due to the introduction of rotation tensor $\boldsymbol{Q}$. We have decomposed the velocity gradient into two parts, in which one should be a rotation tensor.

As a consequence of Eq. (17), if set $\boldsymbol{K}$ as a symmetric tensor, the skew-symmetric part of the $\boldsymbol{K}$ must vanish, namely, $Skew(\boldsymbol{K}) = (1 - \frac{\sin\omega}{\omega})\boldsymbol{W} = \mathbf{0}$, and we will have a special additive decomposition as follows:

$$\boldsymbol{K} = \boldsymbol{D} - \mathbf{1} - \frac{1}{1+\cos\omega}\boldsymbol{W}^2, \tag{19}$$

$$\boldsymbol{Q} = \mathbf{1} + \boldsymbol{W} + \frac{1}{1+\cos\omega}\boldsymbol{W}^2. \tag{20}$$

Let us go back to the vorticity by substituting the new additive decomposition Eq. (17) in Eq. (2); the vorticity is hence expressed in terms of $\boldsymbol{K}$, $\boldsymbol{Q}$

$$\boldsymbol{\omega} = \varepsilon : (\boldsymbol{K} + \boldsymbol{Q}). \tag{21}$$

Although we have the new additive decomposition $\nabla\boldsymbol{v} = \boldsymbol{K} + \boldsymbol{Q}$, due to the mathematical nature of the permutation tensor, anti-symmetric under the interchange of any two slots,[16] the permutation tensor ε will cancel out the contribution from any symmetric part of the velocity gradient by the double dot product ":".

This can be easily proved by substituting the expressions of both $\boldsymbol{K}$ and $\boldsymbol{Q}$ into Eq. (21)

$$\boldsymbol{\omega} = \varepsilon : \underbrace{[\boldsymbol{D} - \boldsymbol{1} - \frac{1-\cos\omega}{\omega^2}\boldsymbol{W}^2 + (1 - \frac{\sin\omega}{\omega})\boldsymbol{W}]}_{=\boldsymbol{K}} + \varepsilon : \underbrace{[\boldsymbol{1} + \frac{\sin\omega}{\omega}\boldsymbol{W} + \frac{1-\cos\omega}{\omega^2}\boldsymbol{W}^2]}_{=\boldsymbol{Q}}. \quad (22)$$

Notice that $\varepsilon:\boldsymbol{1} = 0$, $\varepsilon:(\boldsymbol{D} - \boldsymbol{1}) = 0$, and $\varepsilon:\boldsymbol{W}^2 = 0$ due to the symmetric nature of tensors of $\boldsymbol{1}$, $\boldsymbol{D}$ and $\boldsymbol{W}^2$,[17] and Eq. (22) is reduced back to Eq. (4), i.e., $\boldsymbol{\omega} = \varepsilon : (1 - \frac{\sin\omega}{\omega})\boldsymbol{W} + \varepsilon : (\frac{\sin\omega}{\omega}\boldsymbol{W}) = \varepsilon : \boldsymbol{W}$, which reveals that the vorticity $\boldsymbol{\omega}$ is only affected by the antisymmetric part of the velocity gradient $\nabla\boldsymbol{v}$ and has nothing to do with its symmetric part $\boldsymbol{D}$.

In summary, we have successfully proposed a new additive decomposition $\nabla\boldsymbol{v} = \boldsymbol{K} + \boldsymbol{Q}$ that is different from Cauchy-Stokes decomposition. The new decomposition might provide a new way of thinking about the vortex vector; however, what is the relationship between the vortex vectors in Refs. 8–13 and the decomposition in Eq. (17) is still an open question and needs for further investigation.

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[16] See http://mathworld.wolfram.com/PermutationTensor.html for permutation symbol.

[17] Let $\boldsymbol{Z} = \boldsymbol{W}^2$, then $\boldsymbol{Z}^T = (\boldsymbol{W}\cdot\boldsymbol{W})^T = \boldsymbol{W}^T\cdot(\boldsymbol{W}^T) = (-\boldsymbol{W})\cdot(-\boldsymbol{W}) = \boldsymbol{W}\cdot\boldsymbol{W} = \boldsymbol{W}^2$, hence $\boldsymbol{Z}^T = \boldsymbol{Z}$.